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A logic-based theory of deductive arguments ☆

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Abstract

We explore a framework for argumentation (based on classical logic) in which an argument is a pair where the first item in the pair is a minimal consistent set of formulae that proves the second item (which is a formula). We provide some basic definitions for arguments, and various kinds of counter-arguments (defeaters). This leads us to the definition of canonical undercuts which we argue are the only defeaters that we need to take into account. We then motivate and formalise the notion of argument trees and argument structures which provide a way of exhaustively collating arguments and counter-arguments. We use argument structures as the basis of our general proposal for argument aggregation.

There are a number of frameworks for modelling argumentation in logic. They incorporate formal representation of individual arguments and techniques for comparing conflicting arguments. In these frameworks, if there are a number of arguments for and against a particular conclusion, an aggregation function determines whether the conclusion is taken to hold. We propose a generalisation of these frameworks. In particular, our new framework makes it possible to define aggregation functions that are sensitive to the number of arguments for or against. We compare our framework with a number of other types of argument systems, and finally discuss an application in reasoning with structured news reports.

Keywords: Argumentation systems; Arguments; Counter-arguments; Inconsistency handling; Logic; Undercuts

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1. Introduction

In an argument, we distinguish the reasons, the conclusion and the method of inference by which the conclusion is meant to follow from the reasons. The nature of inference is diverse and includes analogical inference, causal inference, and inductive inference. We focus on deductive inference and hence on deductive arguments, i.e., the conclusion is a deductively valid consequence of the reasons. We investigate the formalisation of such arguments in the setting of classical logic. So, our starting position is that a deductive argument consists of a claim entailed by a collection of statements such that the claim as well as the statements are denoted by formulae of classical logic and entailment is identified with deduction in classical logic.

In our framework, an argument is simply a pair where the first item in the pair is a minimal consistent set of formulae that proves the second item. That is, we account for the reasons and the conclusion of an argument though we do not indicate the method of inference since it does not differ from one argument to another: We only consider deductive arguments, hence the method of inference for each and every argument is always entailment according to classical logic.

Most proposals for modelling argumentation in logic are very limited in the way that they combine arguments for and against a particular conclusion following. A simple form of argumentation is that a conclusion follows if and only if there is an argument for the conclusion and no argument against the conclusion. In our approach, we check how each argument is challenged by other arguments, and by recursion for these counter-arguments. Technically, an argument is undercut if and only if some of the reasons for the argument are rebutted. Each undercut to a counter-argument is itself an argument and so may be undercut, and so by recursion each undercut needs to be considered. Exploring systematically the universe of arguments in order to present an exhaustive synthesis of the relevant chains of undercuts for a given argument is the basic principle of our approach.

Following this, our notion of an argument tree is that it is a synthesis of all the arguments that challenge the argument at the root of the tree, and it also contains all counter-arguments that challenge these arguments and so on recursively.

Modelling argumentation has been a subject of research as long as the study of logic. They are closely intertwined topics, and modelling argumentation in logic is a natural, and important, research goal. A useful introduction to argumentation is in [23], and comprehensive recent reviews of argumentation in logic include [5,20]. The argumentation formalism that we give in this paper, including the notions of argument trees and argument structures, provides a complementary addition to the set of existing proposals for logic-based argumentation systems.

In Sections 2, 3, and 4, we provide some basic definitions for arguments, and various kinds of defeaters. This leads us to the definition of canonical undercuts which we argue are the only defeaters that we need to take into account (Section 5). We then motivate and formalise the notion of argument trees and argument structures which are a way of exhaustively collating arguments and counter-arguments (Sections 6–8). We use argument structures as the basis of our general proposal for argument aggregation. In Section 9, we compare our framework with some other logic-based argument systems. Finally, we discuss an application to argumentation with structured news report (Section 10).

2. Preliminaries

We assume familiarity with classical logic.

We consider a propositional language. We use $\alpha, \beta, \gamma, \ldots$ to denote formulae and $\Delta, \Phi, \Psi, \ldots$ to denote sets of formulae. Deduction in classical propositional logic is denoted by the symbol $\vdash$ as usual. In addition, $\perp$ is used to denote inconsistency. So, $\Phi \nvdash \perp$ for example means that Φ is consistent.

For the following definitions, we first assume a database Δ (a finite set of formulae) and use this Δ throughout.

We further assume that every subset of Δ is given an enumeration $\langle\alpha_1, \ldots, \alpha_n\rangle$ of its elements, which we call its canonical enumeration. This really is not a demanding constraint: In particular, the constraint is satisfied whenever we impose an arbitrary total ordering over Δ. Importantly, the order has no meaning and is not meant to represent any respective importance of formulae in Δ. It is only a convenient way to indicate the order in which we assume the formulae in any subset of Δ are conjoined to make a formula logically equivalent to that subset.

The paradigm for our approach is a large repository of information, represented by Δ, from which arguments can be constructed for and against arbitrary claims. Apart from information being understood as declarative statements, there is no a priori restriction on the contents, and the pieces of information in the repository can be as complex as possible. Therefore, Δ is not expected to be consistent. It need even not be the case that every single formula in Δ is consistent.

3. Arguments

Here we adopt a very common intuitive notion of an argument and consider some of the ramifications of the definition. Essentially, an argument is a set of relevant formulae that can be used to classically prove some point, together with that point (we represent a point by a formula).

Definition 3.1. An *argument* is a pair $\langle\Phi, \alpha\rangle$ such that

(1) $\Phi \nvdash \perp$.
(2) $\Phi \vdash \alpha$.
(3) Φ is a minimal subset of Δ satisfying (2).

We say that $\langle\Phi, \alpha\rangle$ is an argument for α. We call α the consequent of the argument and Φ the support of the argument.

The minimality condition is not an absolute requirement, although some properties below depend on it. Importantly, the condition is not of a mere technical nature. The underlying idea is that an argument makes explicit the connection between reasons for a claim and the claim itself. But that would not be the case if the reasons were not exactly identified. In other words, if reasons incorporated irrelevant information and so included formulas not used in the proof of the conclusion.

Example 3.2. Let $\Delta = \{\alpha, \alpha \to \beta, \gamma \to \neg\beta, \gamma, \delta, \delta \to \beta, \neg\alpha, \neg\gamma\}$. Some arguments are:

$\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$

$\langle\{\gamma \to \neg\beta, \gamma\}, \neg\beta\rangle$

$\langle\{\delta, \delta \to \beta\}, \beta\rangle$

$\langle\{\neg\alpha\}, \neg\alpha\rangle$

$\langle\{\neg\gamma\}, \neg\gamma\rangle$

$\langle\{\alpha \to \beta\}, \neg\alpha \vee \beta\rangle$

$\langle\{\neg\gamma\}, \delta \to \neg\gamma\rangle$

Arguments are not independent. In a sense, some encompass others (possibly up to some form of equivalence). To clarify this requires a few definitions as follows.

Definition 3.3. An argument $\langle\Phi, \alpha\rangle$ is *more conservative* than an argument $\langle\Psi, \beta\rangle$ iff $\Phi \subseteq \Psi$ and $\beta \vdash \alpha$.

Example 3.4. $\langle\{\alpha\}, \alpha \vee \beta\rangle$ is more conservative than $\langle\{\alpha, \alpha \to \beta\}, \beta\rangle$. Here, the latter argument can be obtained from the former (using $\alpha \to \beta$ as an extra hypothesis) but the reader is warned that this is not the case in general as we now discuss.

Example 3.4 suggests that an argument $\langle\Psi, \beta\rangle$ can be obtained from a more conservative argument $\langle\Phi, \alpha\rangle$ by using $\Psi \setminus \Phi$ together with α in order to deduce β (in symbols, $\{\alpha\} \cup \Psi \setminus \Phi \vdash \beta$ or equivalently, $\Psi \setminus \Phi \vdash \alpha \to \beta$). As just mentioned, this does not hold in full generality. A counterexample consists of $\langle\{\alpha \wedge \gamma\}, \alpha\rangle$ and $\langle\{\alpha \wedge \gamma, \neg\alpha \vee \beta \vee \neg\gamma\}, \beta\rangle$. However, a weaker property holds:

Theorem 3.5. *If $\langle\Phi, \alpha\rangle$ is more conservative than $\langle\Psi, \beta\rangle$ then $\Psi \setminus \Phi \vdash \phi \to (\alpha \to \beta)$ for some formula ϕ such that $\Phi \vdash \phi$ and $\phi \nvdash \alpha$ unless α is a tautology.*

Proof. Since $\langle\Phi, \alpha\rangle$ is an argument, Φ is finite and $\Phi \vdash \alpha$ so that Φ is logically equivalent to $\alpha \wedge (\alpha \to \phi')$ for some formula ϕ' (that is, Φ is logically equivalent to $\alpha \wedge \phi$ where ϕ is $\alpha \to \phi'$). Since $\langle\Phi, \alpha\rangle$ is more conservative than $\langle\Psi, \beta\rangle$, $\Psi = \Phi \cup \Psi \setminus \Phi$. Since $\langle\Psi, \beta\rangle$ is an argument, $\Psi \vdash \beta$. Hence, $\Phi \cup \Psi \setminus \Phi \vdash \beta$. Then, $\{\alpha \wedge \phi\} \cup \Psi \setminus \Phi \vdash \beta$. It follows that $\Psi \setminus \Phi \vdash \phi \to (\alpha \to \beta)$. There only remains to show that $\Phi \vdash \phi$ (which is trivial) and that $\phi \nvdash \alpha$ unless α is a tautology. Assuming $\phi \vdash \alpha$ gives $\alpha \to \phi' \vdash \alpha$, but the latter means that α is a tautology. □

The interesting case, as in Example 3.4, is when ϕ can be a tautology.

Theorem 3.6. *Being more conservative defines a pre-ordering over arguments. Minimal arguments always exist, unless all formulas in Δ are inconsistent. Maximal arguments always exist*: *They are $\langle\emptyset, \top\rangle$ where $\top$ is any tautology.*

Proof. Reflexivity and transitivity result from the fact that these two properties are satisfied by set inclusion and logical consequence.

Easy as well is the case of $\langle\emptyset, \top\rangle$: An argument is maximal iff it is of this form because $\emptyset$ and $\top$ are extremal with respect to set inclusion and logical consequence, respectively.

Assuming that some formula in Δ is consistent, Δ has at least one maximal consistent subset Θ. Since Δ is finite, so is Θ and there exists a formula α that Θ is logically equivalent with. Also, there is some minimal $\Phi \subseteq \Theta \subseteq \Delta$ such that Φ and Θ are logically equivalent. Clearly, Φ is consistent and Φ is a minimal subset of Δ such that $\Phi \vdash \alpha$. In other words, $\langle\Phi, \alpha\rangle$ is an argument. There only remains to show that it is minimal. Consider an argument $\langle\Psi, \beta\rangle$ such that $\langle\Phi, \alpha\rangle$ is more conservative than $\langle\Psi, \beta\rangle$. According to Definition 3.3, $\Phi \subseteq \Psi$ and $\beta \vdash \alpha$. Since Φ is logically equivalent with a maximal consistent subset of Δ, it follows that Ψ is logically equivalent with Φ (because Ψ is a consistent subset of Δ by definition of an argument). So, α is logically equivalent with each of Φ and Ψ. As a consequence, $\Psi \vdash \beta$ and $\beta \vdash \alpha$ yield that α is logically equivalent with β, too. Since Φ is a minimal subset of Δ such that $\Phi \vdash \alpha$ (cf above), it follows that Φ is a minimal subset of Δ such that $\Phi \vdash \beta$. However, Ψ is also a minimal subset of Δ such that $\Psi \vdash \beta$ (by definition of an argument). Hence, $\Phi = \Psi$ (due to $\Phi \subseteq \Psi$). In all, $\langle\Psi, \beta\rangle$ is more conservative than $\langle\Phi, \alpha\rangle$. Stated otherwise, we have just shown that if $\langle\Phi, \alpha\rangle$ is more conservative than $\langle\Psi, \beta\rangle$ then the converse is true as well. I.e., $\langle\Phi, \alpha\rangle$ is minimal with respect to being more conservative (as applied to arguments). $\square$

A useful notion is then that of a normal form (a function such that any formula is mapped to a logically equivalent formula and, if understood in a strict sense as here, such that any two logically equivalent formulas are mapped to the same formula).

Theorem 3.7. *Given a normal form, being more conservative defines an ordering provided that only arguments which have a consequent in normal form are considered. The ordered set of all such arguments is an upper semilattice* (*when restricted to the language of* Δ). *The greatest argument always exists, it is* $\langle\emptyset, \top\rangle$.

Proof. Let $\langle\Phi, \alpha\rangle$ and $\langle\Psi, \beta\rangle$ be more conservative than each other. Clearly, $\Phi = \Psi$. Also, $\alpha \vdash \beta$ and $\beta \vdash \alpha$. That is, α and β are logically equivalent. Since α and β are in normal form, $\alpha = \beta$. So, antisymmetry holds while reflexivity and transitivity follow from Theorem 3.6.

Since Δ is finite, there only are finitely many arguments $\langle\Omega, \gamma\rangle$ for each γ. Also, the language of Δ contains only a finite number of atomic formulae and it allows only for a finite number of formulas that are not logically equivalent. That is, there only is a finite number of arguments if they are restricted to the language of Δ. So, all the upper bounds of two given arguments $\langle\Phi, \alpha\rangle$ and $\langle\Psi, \beta\rangle$ form a finite set $\{\langle\Omega_1, \gamma_1\rangle, \ldots, \langle\Omega_n, \gamma_n\rangle\}$. For $i = 1, \ldots, n$, $\Omega_i \subseteq \Phi$ and $\Omega_i \subseteq \Psi$. Hence, $\Omega_i \subseteq \Phi \cap \Psi$. Then, $\Phi \cap \Psi \vdash \gamma_i$ because $\langle\Omega_i, \gamma_i\rangle$ is an argument. It follows that $\Phi \cap \Psi \vdash \gamma_1 \wedge \cdots \wedge \gamma_n$ and an argument $\langle\Theta, \gamma_1 \wedge \cdots \wedge \gamma_n\rangle$ can be constructed where Θ is a minimal subset of $\Phi \cap \Psi$ such that $\Theta \vdash \gamma_1 \wedge \cdots \wedge \gamma_n$ (since Φ and Ψ are consistent by definition of an argument, Θ is consistent). For matter of convenience, $\gamma_1 \wedge \cdots \wedge \gamma_n$ is assumed to be in normal form as doing so obviously causes no harm here. Clearly, every $\langle\Omega_i, \gamma_i\rangle$ is more conservative than $\langle\Theta, \gamma_1 \wedge \cdots \wedge \gamma_n\rangle$. This means that $\langle\Theta, \gamma_1 \wedge \cdots \wedge \gamma_n\rangle$ is the l.u.b. of $\langle\Phi, \alpha\rangle$ and $\langle\Psi, \beta\rangle$ (uniqueness of Θ is imposed by the usual property of orderings of allowing for at most one l.u.b. of any subset).

As for the greatest argument, it is trivial that $\emptyset \subseteq \Phi$ and $\alpha \vdash \top$ for all Φ and α. □

Example 3.8. The g.l.b. of $\langle\{\alpha \wedge \beta\}, \alpha\rangle$ and $\langle\{\alpha \wedge \neg\beta\}, \alpha\rangle$ does not exist. If $\Delta = \{\alpha \wedge \beta, \alpha \wedge \neg\beta\}$, then there is no least argument. Taking now $\Delta = \{\alpha, \beta, \alpha \leftrightarrow \beta\}$, there is no least argument either (although Δ is consistent). Even though $\Delta = \{\alpha, \beta \wedge \neg\beta\}$ is inconsistent, the least argument exists: $\langle\{\alpha\}, \alpha'\rangle$ (where α' stands for the normal form of α). As the last illustration, $\Delta = \{\alpha \vee \beta, \beta\}$ admits the least argument $\langle\{\beta\}, \beta'\rangle$ (where β' stands for the normal form of β).

In any case, $\langle\emptyset, \top\rangle$ is more conservative than any other argument.

Also, irrespective of whether we have an ordering or not, the "being more conservative" relation induces, as any pre-ordering does, an equivalence relation (linking any two arguments that are more conservative than each other). However, another basis for equating two arguments with each other comes to mind: Pairwise logical equivalence of the components of both arguments.

Definition 3.9. Two arguments $\langle\Phi, \alpha\rangle$ and $\langle\Psi, \beta\rangle$ are *equivalent* iff Φ is logically equivalent to Ψ and α is logically equivalent to β.

Theorem 3.10. *Two arguments are equivalent whenever each is more conservative than the other. In partial converse, if two arguments are equivalent then either each is more conservative than the other or neither is.*

Proof. We only prove the second part. Consider two equivalent arguments $\langle\Phi, \alpha\rangle$ and $\langle\Psi, \beta\rangle$ such that $\langle\Phi, \alpha\rangle$ is more conservative than $\langle\Psi, \beta\rangle$. Of course, $\Phi \vdash \alpha$. According to the definition of equivalent arguments, α is logically equivalent with β. So, $\Phi \vdash \beta$. By definition of an argument, Ψ is a minimal subset of Δ satisfying $\Psi \vdash \beta$. Hence, $\Psi = \Phi$ because $\Phi \subseteq \Psi$ follows from the fact that $\langle\Phi, \alpha\rangle$ is more conservative than $\langle\Psi, \beta\rangle$. Finally, $\Psi = \Phi$ and α being logically equivalent with β make each of $\langle\Psi, \beta\rangle$ and $\langle\Phi, \alpha\rangle$ to be more conservative than the other. □

So, there exist equivalent arguments $\langle\Phi, \alpha\rangle$ and $\langle\Psi, \beta\rangle$ that fail to be more conservative than each other (as in Example 3.11 below). However, if $\langle\Phi, \alpha\rangle$ is strictly more conservative than $\langle\Psi, \beta\rangle$ (meaning that $\langle\Phi, \alpha\rangle$ is more conservative than $\langle\Psi, \beta\rangle$ but $\langle\Psi, \beta\rangle$ is not more conservative than $\langle\Phi, \alpha\rangle$) then $\langle\Phi, \alpha\rangle$ and $\langle\Psi, \beta\rangle$ are not equivalent.

Example 3.11. Let $\Phi = \{\alpha, \beta\}$ and $\Psi = \{\alpha \vee \beta, \alpha \leftrightarrow \beta\}$. The arguments $\langle\Phi, \alpha \wedge \beta\rangle$ and $\langle\Psi, \alpha \wedge \beta\rangle$ are equivalent even though none is more conservative than the other. This means that there exist two distinct subsets of Δ (namely, Φ and Ψ) supporting $\alpha \wedge \beta$.

Whilst equivalent arguments make the same point (that is, the same inference), we do want to distinguish equivalent arguments from each other. What we do not want is to distinguish between arguments that are more conservative than each other.

4. Defeaters, rebuttals and undercuts

In recent literature [5,20], conflict among arguments is investigated in more or less abstract forms. The most abstract form is that of an attack relation, defined as a subset of the cartesian product of the set of all arguments with itself [7]. Exploring such an approach where the internal structure of arguments is ignored, thus opposing our view that an argument comes with a claim, is delayed to Section 9. A more concrete form of conflict is captured with the notion of defeaters, which are arguments whose conclusion refutes the support of another argument [17–19,24,25]. This gives us a general way for an argument to challenge another.

Definition 4.1. A *defeater* for an argument $\langle \Phi, \alpha \rangle$ is an argument $\langle \Psi, \beta \rangle$ such that $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ for some $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$.

Example 4.2. Let $\Delta = \{\neg\alpha, \alpha \vee \beta, \alpha \leftrightarrow \beta, \gamma \rightarrow \alpha\}$. Then, $\langle \{\alpha \vee \beta, \alpha \leftrightarrow \beta\}, \alpha \wedge \beta \rangle$ is a defeater for $\langle \{\neg\alpha, \gamma \rightarrow \alpha\}, \neg\gamma \rangle$. A more conservative defeater for $\langle \{\neg\alpha, \gamma \rightarrow \alpha\}, \neg\gamma \rangle$ is $\langle \{\alpha \vee \beta, \alpha \leftrightarrow \beta\}, \alpha \vee \gamma \rangle$.

Some arguments directly oppose the support of others, which amounts to the notion of an undercut.

Definition 4.3. An *undercut* for an argument $\langle \Phi, \alpha \rangle$ is an argument $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ where $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$.

Example 4.4. Let $\Delta = \{\alpha, \alpha \rightarrow \beta, \gamma, \gamma \rightarrow \neg\alpha\}$. Then, $\langle \{\gamma, \gamma \rightarrow \neg\alpha\}, \neg(\alpha \wedge (\alpha \rightarrow \beta)) \rangle$ is an undercut for $\langle \{\alpha, \alpha \rightarrow \beta\}, \beta \rangle$. A less conservative undercut for $\langle \{\alpha, \alpha \rightarrow \beta\}, \beta \rangle$ is $\langle \{\gamma, \gamma \rightarrow \neg\alpha\}, \neg\alpha \rangle$.

Presumably, the most direct form of a conflict between arguments is when two arguments have opposite conclusions. This case is captured in the literature through the notion of a rebuttal.

Definition 4.5. An argument $\langle \Psi, \beta \rangle$ is a *rebuttal* for an argument $\langle \Phi, \alpha \rangle$ iff $\beta \leftrightarrow \neg\alpha$ is a tautology.

Trivially, undercuts are defeaters but the next result shows that rebuttals too are defeaters.

Theorem 4.6. *If $\langle \Psi, \beta \rangle$ is a rebuttal for an argument $\langle \Phi, \alpha \rangle$ then $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$.*

Proof. By definition of an argument, $\Phi \vdash \alpha$. By classical logic, $\neg\alpha \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ where $\Phi = \{\phi_1, \ldots, \phi_n\}$. As $\beta \leftrightarrow \neg\alpha$ is a tautology, $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ follows. □

Theorem 4.7. *Let $\langle \Psi, \beta \rangle$ be a defeater for an argument $\langle \Phi, \alpha \rangle$. If $\alpha \vee \beta$ is a tautology and $\{\alpha, \beta\} \vdash \phi$ for each $\phi \in \Phi$ then $\langle \Psi, \beta \rangle$ is a rebuttal for $\langle \Phi, \alpha \rangle$.*

Proof. Since $\langle \Psi, \beta \rangle$ is a defeater of $\langle \Phi, \alpha \rangle$, we have $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ for some $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$. By assumption, $\{\alpha, \beta\} \vdash \phi_i$ for $i = 1, \ldots, n$. Therefore, $\{\alpha, \beta\} \vdash \bot$. Using classical logic, it follows that $\beta \leftrightarrow \neg\alpha$ is a tautology because $\alpha \vee \beta$ is a tautology (cf. the assumptions). □

Of course, not all defeaters can meet the conditions in Theorem 4.7 and it may happen that an argument has defeaters but no rebuttals as illustrated now:

Example 4.8. Let $\Delta = \{\alpha \wedge \beta, \neg\beta\}$. Then, $\langle \{\alpha \wedge \beta\}, \alpha \rangle$ has at least one defeater but no rebuttal.

Evidently, an undercut for an argument need not be a rebuttal for that argument. Importantly, a rebuttal for an argument need not be an undercut for that argument. However, a rebuttal for an argument is a less conservative version of a specific undercut for that argument as we now prove.

Theorem 4.9. *If $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$ then there exists an undercut for $\langle \Phi, \alpha \rangle$ which is more conservative than $\langle \Psi, \beta \rangle$.*

Proof. By definition of a defeater, $\Psi \nvdash \bot$ and $\Psi \vdash \beta$ while $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ for some $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$. There then exists a minimal subset $\Psi' \subseteq \Psi \subseteq \Delta$ such that $\Psi' \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Of course, $\Psi' \nvdash \bot$. Therefore, $\langle \Psi', \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ is an argument and it clearly is an undercut for $\langle \Phi, \alpha \rangle$. Verification that $\langle \Psi', \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ is more conservative than $\langle \Psi, \beta \rangle$ is immediate. □

Corollary 4.10. *If $\langle \Psi, \beta \rangle$ is a rebuttal for $\langle \Phi, \alpha \rangle$ then there exists an undercut for $\langle \Phi, \alpha \rangle$ which is more conservative than $\langle \Psi, \beta \rangle$.*

The undercut mentioned in Theorem 4.9 and Corollary 4.10 is strictly more conservative than $\langle \Psi, \beta \rangle$ whenever $\neg\beta$ fails to be logically equivalent with Φ.

A phenomenon similar to what Corollary 4.10 describes occurs in argument structures (cf. Theorem 8.8).

As a first illustration, $\langle \{\neg\alpha\}, \neg\alpha \rangle$ is an undercut for $\langle \{\alpha, \alpha \to \beta\}, \beta \rangle$ but is not a rebuttal for it. Clearly, $\langle \{\neg\alpha\}, \neg\alpha \rangle$ does not rule out β. Actually, an undercut may even agree with the conclusion of the objected argument: $\langle \{\beta \wedge \neg\alpha\}, \neg\alpha \rangle$ is an undercut for $\langle \{\alpha, \alpha \to \beta\}, \beta \rangle$. In this case, we have an argument with an undercut that conflicts with the support of the argument but implicitly provides an alternative way to derive the conclusion of the argument. This should make it clear that an undercut need not question the conclusion of an argument but only the reason(s) given by that argument to support its conclusion. Of course, there are also undercuts that challenge an argument on both counts: Just consider $\langle \{\neg\alpha \wedge \neg\beta\}, \neg\alpha \rangle$ which is such an undercut for $\langle \{\alpha, \alpha \to \beta\}, \beta \rangle$.

As another example, $\langle \{\neg\beta\}, \neg\beta \rangle$ is a rebuttal for $\langle \{\alpha, \alpha \to \beta\}, \beta \rangle$ but is not an undercut for it because β is not in $\{\alpha, \alpha \to \beta\}$. Observe that there is not even an argument equivalent to $\langle \{\neg\beta\}, \neg\beta \rangle$ which would be an undercut for $\langle \{\alpha, \alpha \to \beta\}, \beta \rangle$: In order to be an undercut for $\langle \{\alpha, \alpha \to \beta\}, \beta \rangle$, an argument should be of the form $\langle \Phi, \neg\alpha \rangle$, $\langle \Phi, \neg(\alpha \to \beta) \rangle$ or

$\langle \Phi, \neg(\alpha \wedge (\alpha \to \beta)) \rangle$ but $\neg\beta$ is not logically equivalent to $\neg\alpha$, $\neg(\alpha \to \beta)$ or $\neg(\alpha \wedge (\alpha \to \beta))$.

Theorem 4.11. *If an argument has a defeater, then there exists an undercut for its defeater.*

Proof. Let $\langle \Psi, \beta \rangle$ be a defeater for $\langle \Phi, \alpha \rangle$. That is, $\Psi \vdash \beta$ and $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ for some $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$. Writing Ψ as $\{\psi_1, \ldots, \psi_m\}$, we get $\{\psi_1, \ldots, \psi_m\} \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. By classical logic, $\{\phi_1, \ldots, \phi_n\} \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_m)$ and $\Phi \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_m)$. Let $\Phi' \subseteq \Phi$ be a minimal subset entailing $\neg(\psi_1 \wedge \cdots \wedge \psi_m)$. Then, $\langle \Phi', \neg(\psi_1 \wedge \cdots \wedge \psi_m) \rangle$ is an argument and it is an undercut for $\langle \Psi, \beta \rangle$. $\square$

Corollary 4.12. *If an argument A has at least one defeater, then there exists an infinite sequence of arguments $(A_n)_{n \in \omega^*}$ such that A_1 is A and A_{n+1} is an undercut of A_n for every $n \in \omega^*$.*

The above Corollary 4.12 is obviously a potential concern for representing and comparing arguments. We address this question in Section 6.

Theorem 4.13. *Let $\langle \Phi, \alpha \rangle$ be an argument for which $\langle \Psi, \beta \rangle$ is a defeater. Then, $\Psi \nsubseteq \Phi$.*

Proof. Assume the contrary, $\Psi \subseteq \Phi$. By definition of a defeater, $\Psi \vdash \beta$ and $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ for some $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$. Therefore, $\Psi \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Since $\Psi \subseteq \Phi$, monotonicity then yields $\Phi \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. However, $\Phi \vdash \phi_1 \wedge \cdots \wedge \phi_n$ because $\Phi \supseteq \{\phi_1, \ldots, \phi_n\}$. That is, $\Phi \vdash \perp$ and this contradicts the assumption that $\langle \Phi, \alpha \rangle$ is an argument. $\square$

Theorem 4.13 proves that, in the sense of Definition 3.1 and Definition 4.1, no argument is self-defeating.

Theorem 4.14. *Given two arguments $\langle \Phi, \alpha \rangle$ and $\langle \Psi, \beta \rangle$ such that $\{\alpha, \beta\} \vdash \phi$ for each $\phi \in \Phi$, if $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$, then $\langle \Phi, \alpha \rangle$ is a defeater for $\langle \Psi, \beta \rangle$.*

Proof. By assumption, $\Psi \vdash \beta$ and $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ for some $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$. By classical logic, it follows that $\Psi \cup \Phi \vdash \perp$. So, $\Psi \cup \{\alpha\} \vdash \perp$ because $\Psi \vdash \beta$ and $\{\alpha, \beta\}$ entails Φ. Hence, $\alpha \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_m)$ where $\{\psi_1, \ldots, \psi_m\} = \Psi$. $\square$

Corollary 4.15. *Let α be logically equivalent with Φ. If $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$, then $\langle \Phi, \alpha \rangle$ is a defeater for $\langle \Psi, \beta \rangle$.*

Proof. Since $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$, it follows that $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ for some $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$. By classical logic, $\Phi \cup \{\beta\} \vdash \perp$ and thus $\{\alpha, \beta\} \vdash \perp$ because α is logically equivalent with Φ. By classical logic, $\{\alpha, \beta\} \vdash \phi$ for each $\phi \in \Phi$ and Theorem 4.14 applies. $\square$

Corollary 4.16. *If $\langle \Psi, \beta \rangle$ is a rebuttal for $\langle \Phi, \alpha \rangle$, then $\langle \Phi, \alpha \rangle$ is a rebuttal for $\langle \Psi, \beta \rangle$.*

Proof. As $\langle \Psi, \beta \rangle$ is a rebuttal for $\langle \Phi, \alpha \rangle$, it follows that $\alpha \vee \beta$ is a tautology and $\{\alpha, \beta\} \vdash \psi$ for each $\psi \in \Psi$. Apply Theorem 4.6 to the assumption, then apply Theorem 4.14, and, in view of what we just proved, apply Theorem 4.7. [1] □

Theorem 4.17. *Given two arguments $\langle \Phi, \alpha \rangle$ and $\langle \Psi, \beta \rangle$ such that $\neg(\alpha \wedge \beta)$ is a tautology, $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$ and $\langle \Phi, \alpha \rangle$ is a defeater for $\langle \Psi, \beta \rangle$.*

Proof. As $\langle \Phi, \alpha \rangle$ is an argument, $\Phi \vdash \alpha$. By assumption, $\{\alpha, \beta\} \vdash \bot$. Therefore, $\Phi \cup \{\beta\} \vdash \bot$. That is, $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ where $\{\phi_1, \ldots, \phi_n\} = \Phi$. The other case is clearly symmetric. □

While Theorem 4.13 expresses that the defeat relation is antireflexive, Theorem 4.14 and Theorem 4.17 show that the defeat relation is symmetric on various parts of the domain.

Theorem 4.18. *Δ is inconsistent if there exists an argument that has at least one defeater. Should there be some inconsistent formula in Δ, the converse is untrue. When no formula in Δ is inconsistent, the converse is true in the form: If Δ is inconsistent then there exists an argument that has at least one rebuttal.*

Proof. Suppose $\langle \Psi, \beta \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$. Hence, there exists $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$ such that $\Psi \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. By classical logic, $\Psi \cup \{\phi_1, \ldots, \phi_n\} \vdash \bot$ and $\Psi \cup \Phi \vdash \bot$. Since $\Psi \cup \Phi \subseteq \Delta$, we have $\Delta \vdash \bot$. As for the converse, if each formula in Δ is consistent and Δ is inconsistent then there exists a minimal inconsistent subset Φ. That is, $\Phi \vdash \bot$. By classical logic, $\Phi \setminus \{\phi\} \vdash \phi \to \bot$ for any $\phi \in \Phi$. I.e., $\Phi \setminus \{\phi\} \vdash \neg\phi$. Clearly, $\{\phi\}$ and $\Phi \setminus \{\phi\}$ are consistent. Also, there exists a minimal subset Ψ of $\Phi \setminus \{\phi\}$ such that $\Psi \vdash \neg\phi$. So, $\langle \{\phi\}, \phi \rangle$ and $\langle \Psi, \neg\phi \rangle$ are arguments. Of course, $\langle \Psi, \neg\phi \rangle$ is a rebuttal for $\langle \{\phi\}, \phi \rangle$. □

Corollary 4.19. *Δ is inconsistent if there exists an argument that has at least one undercut. The converse is true when each formula in Δ is consistent.*

As arguments can be ordered from more conservative to less conservative, there is a clear and unambiguous notion of maximally conservative defeaters for a given argument (the ones which are representative of all defeaters for that argument):

Definition 4.20. $\langle \Psi, \beta \rangle$ is a *maximally conservative defeater* of $\langle \Phi, \alpha \rangle$ iff $\langle \Psi, \beta \rangle$ is a defeater of $\langle \Phi, \alpha \rangle$ such that no defeaters of $\langle \Phi, \alpha \rangle$ are strictly more conservative than $\langle \Psi, \beta \rangle$ (that is, for all defeaters $\langle \Psi', \beta' \rangle$ of $\langle \Phi, \alpha \rangle$, if $\Psi' \subseteq \Psi$ and $\beta \vdash \beta'$ then $\Psi \subseteq \Psi'$ and $\beta' \vdash \beta$).

Theorem 4.21. *Let $\langle \Psi, \beta \rangle$ be a maximally conservative defeater for an argument $\langle \Phi, \alpha \rangle$. Then, $\langle \Psi, \gamma \rangle$ is a maximally conservative defeater for $\langle \Phi, \alpha \rangle$ iff γ is logically equivalent with β.*

[1] It is also possible to prove the result directly, in the obvious way.

Proof. We prove the non-trivial part. Let $\langle \Psi, \beta \rangle$ and $\langle \Psi, \gamma \rangle$ be maximally conservative defeaters for $\langle \Phi, \alpha \rangle$. Applying classical logic on top of the definition of a defeater, $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ and $\gamma \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ where $\Phi = \{\phi_1, \ldots, \phi_n\}$. So, $\beta \vee \gamma \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Now, there exists some minimal $\Psi' \subseteq \Psi$ such that $\Psi' \vdash \beta \vee \gamma$ (as $\Psi \vdash \beta$ and $\Psi \vdash \gamma$). Moreover, $\Psi' \nvdash \bot$ because $\Psi \nvdash \bot$ by definition of a defeater (which is required to be an argument). Hence, $\langle \Psi', \beta \vee \gamma \rangle$ is an argument and it is a defeater for $\langle \Phi, \alpha \rangle$ as we have already proven $\beta \vee \gamma \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Due to $\Psi' \subseteq \Psi$, it follows that $\langle \Psi', \beta \vee \gamma \rangle$ is more conservative than $\langle \Psi, \beta \rangle$ and $\langle \Psi, \gamma \rangle$. Since each of these two is a maximally conservative defeater for $\langle \Phi, \alpha \rangle$, we obtain $\beta \vee \gamma \vdash \beta$ and $\beta \vee \gamma \vdash \gamma$. That is, β and γ are logically equivalent. $\square$

Notice that Theorem 4.21 does not extend to undercuts because they are syntax-dependent (in an undercut, the consequent is always a formula governed by negation).

Theorem 4.22. *If $\langle \Psi, \beta \rangle$ is a maximally conservative defeater of $\langle \Phi, \alpha \rangle$ then $\langle \Psi, \beta' \rangle$ is an undercut of $\langle \Phi, \alpha \rangle$ for some β' which is logically equivalent with β.*

Proof. Let $\langle \Psi, \beta \rangle$ be a defeater for $\langle \Phi, \alpha \rangle$ such that for all defeaters $\langle \Psi', \beta' \rangle$ of $\langle \Phi, \alpha \rangle$, if $\Psi' \subseteq \Psi$ and $\beta \vdash \beta'$ then $\Psi \subseteq \Psi'$ and $\beta' \vdash \beta$. By definition of a defeater, $\beta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ for some $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$ while $\Psi \vdash \beta$ and $\Psi \nvdash \bot$. Then, $\langle \Psi', \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ is clearly an argument whenever Ψ' is taken to denote a minimal subset of Ψ such that $\Psi' \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. It is an undercut for $\langle \Phi, \alpha \rangle$. Therefore, it is a defeater for $\langle \Phi, \alpha \rangle$. Applying the assumption stated at the start of the proof, $\Psi = \Psi'$ and $\neg(\phi_1 \wedge \cdots \wedge \phi_n) \vdash \beta$. So, β is logically equivalent with $\neg(\phi_1 \wedge \cdots \wedge \phi_n)$ while $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ is an undercut for $\langle \Phi, \alpha \rangle$. $\square$

Theorem 4.22 suggests to focus on undercuts when synthetizing counter-arguments to a given argument as is investigated from now on.

5. Canonical undercuts

As already defined above, an undercut for an argument $\langle \Phi, \alpha \rangle$ is an argument $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ where $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$ and $\Phi \cup \Psi \subseteq \Delta$ by definition of an argument.

While Theorem 4.9 and Theorem 4.22 point to undercuts as candidates to be representative of all defeaters for an argument, maximally conservative undercuts are even better candidates.

Definition 5.1. $\langle \Psi, \beta \rangle$ is a *maximally conservative undercut* of $\langle \Phi, \alpha \rangle$ iff $\langle \Psi, \beta \rangle$ is an undercut of $\langle \Phi, \alpha \rangle$ such that no undercuts of $\langle \Phi, \alpha \rangle$ are strictly more conservative than $\langle \Psi, \beta \rangle$ (that is, for all undercuts $\langle \Psi', \beta' \rangle$ of $\langle \Phi, \alpha \rangle$, if $\Psi' \subseteq \Psi$ and $\beta \vdash \beta'$ then $\Psi \subseteq \Psi'$ and $\beta' \vdash \beta$).

Notice that the consequent of a maximally conservative undercut for an argument is exactly the negation of the full support of the argument:

Theorem 5.2. *If $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is a maximally conservative undercut for an argument $\langle \Phi, \alpha\rangle$, then $\Phi = \{\phi_1, \ldots, \phi_n\}$.*

Proof. Of course, $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$. Assume that there exists ϕ such that $\phi \in \Phi \setminus \{\phi_1, \ldots, \phi_n\}$. Since $\langle \Phi, \alpha\rangle$ is an argument, Φ is a minimal subset of Δ such that $\Phi \vdash \alpha$. Hence, $\{\phi_1, \ldots, \phi_n\} \nvdash \phi$ and $\neg(\phi \wedge \phi_1 \wedge \cdots \wedge \phi_n) \nvdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Now, $\Psi \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Then, $\Psi \vdash \neg(\phi \wedge \phi_1 \wedge \cdots \wedge \phi_n)$. So, there exists $\Psi' \subseteq \Psi$ such that $\langle \Psi', \neg(\phi \wedge \phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is an argument. Since $\neg(\phi_1 \wedge \cdots \wedge \phi_n) \vdash \neg(\phi \wedge \phi_1 \wedge \cdots \wedge \phi_n)$, it follows that $\langle \Psi', \neg(\phi \wedge \phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is more conservative than $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$. In fact, $\langle \Psi', \neg(\phi \wedge \phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is strictly more conservative than $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ because $\neg(\phi \wedge \phi_1 \wedge \cdots \wedge \phi_n) \nvdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Moreover, $\langle \Psi', \neg(\phi \wedge \phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is clearly an undercut for $\langle \Phi, \alpha\rangle$ so that $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ being a maximally conservative undercut for $\langle \Phi, \alpha\rangle$ is contradicted. □

Note that if $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is a maximally conservative undercut for an argument $\langle \Phi, \alpha\rangle$, then so are $\langle \Psi, \neg(\phi_2 \wedge \cdots \wedge \phi_n \wedge \phi_1)\rangle$ and $\langle \Psi, \neg(\phi_3 \wedge \cdots \wedge \phi_n \wedge \phi_1 \wedge \phi_2)\rangle$ and so on. However, they are all identical (in the sense that each is more conservative than the others). We can ignore the unnecessary variants by just considering the canonical undercuts defined as follows.

Definition 5.3. An argument $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is a *canonical undercut* for $\langle \Phi, \alpha\rangle$ iff it is a maximally conservative undercut for $\langle \Phi, \alpha\rangle$ and $\langle \phi_1, \ldots, \phi_n\rangle$ is the canonical enumeration of Φ.

Theorem 5.4. *An argument $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is a canonical undercut for $\langle \Phi, \alpha\rangle$ iff it is an undercut for $\langle \Phi, \alpha\rangle$ and $\langle \phi_1, \ldots, \phi_n\rangle$ is the canonical enumeration of Φ.*

Proof. We prove the non-trivial part. Let $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ be an undercut for $\langle \Phi, \alpha\rangle$ such that $\langle \phi_1, \ldots, \phi_n\rangle$ is the canonical enumeration of Φ. We only need to show that $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is a maximally conservative undercut for $\langle \Phi, \alpha\rangle$. Assume that $\langle \Theta, \neg(\gamma_1 \wedge \cdots \wedge \gamma_m)\rangle$ is an undercut for $\langle \Phi, \alpha\rangle$ that is more conservative than $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$. By Definition 4.3, $\{\gamma_1, \ldots, \gamma_m\} \subseteq \Phi$. Now, $\Phi = \{\phi_1, \ldots, \phi_n\}$. It follows that $\neg(\gamma_1 \wedge \cdots \wedge \gamma_m) \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Hence, $\Theta \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ because $\langle \Theta, \neg(\gamma_1 \wedge \cdots \wedge \gamma_m)\rangle$ is an argument. However, $\Theta \subseteq \Psi$ due to the assumption that $\langle \Theta, \neg(\gamma_1 \wedge \cdots \wedge \gamma_m)\rangle$ is more conservative than $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$. Should Θ be a proper subset of Ψ, it would then be the case that Ψ is not a minimal subset of Δ entailing $\neg(\phi_1 \wedge \cdots \wedge \phi_n)$ and this would contradict the fact that $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is an argument. So, $\Theta = \Psi$ and it follows that $\langle \Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is more conservative than $\langle \Theta, \neg(\gamma_1 \wedge \cdots \wedge \gamma_m)\rangle$ as we already proved $\neg(\gamma_1 \wedge \cdots \wedge \gamma_m) \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. □

Theorem 5.5. *Any two different canonical undercuts for the same argument have the same consequent, but distinct supports.*

Theorem 5.6. *Given two different canonical undercuts for the same argument, none is more conservative than the other.*

Proof. In view of Theorem 5.5, it is enough to consider the case of $\langle \Psi_1, \neg\phi \rangle$ and $\langle \Psi_2, \neg\phi \rangle$ being two different canonical undercuts for the same argument $\langle \Phi, \alpha \rangle$. Assume that $\langle \Psi_1, \neg\phi \rangle$ is more conservative than $\langle \Psi_2, \neg\phi \rangle$. Then, $\langle \Psi_2, \neg\phi \rangle$ is more conservative than $\langle \Psi_1, \neg\phi \rangle$ because $\langle \Psi_1, \neg\phi \rangle$ is an undercut for $\langle \Phi, \alpha \rangle$ and $\langle \Psi_2, \neg\phi \rangle$ is a maximally conservative undercut for $\langle \Phi, \alpha \rangle$. Overall, $\Psi_1 = \Psi_2$. So, $\langle \Psi_1, \neg\phi \rangle = \langle \Psi_2, \neg\phi \rangle$ and this is a contradiction. □

Example 5.7. Let $\Delta = \{\alpha, \beta, \neg\alpha, \neg\beta\}$. Both the following

$$\langle \{\neg\alpha\}, \neg(\alpha \wedge \beta) \rangle$$

$$\langle \{\neg\beta\}, \neg(\alpha \wedge \beta) \rangle$$

are canonical undercuts for $\langle \{\alpha, \beta\}, \alpha \leftrightarrow \beta \rangle$, but neither is more conservative than the other.

Theorem 5.8. *For each defeater $\langle \Psi, \beta \rangle$ of an argument $\langle \Phi, \alpha \rangle$, there exists a canonical undercut for $\langle \Phi, \alpha \rangle$ that is more conservative than $\langle \Psi, \beta \rangle$.*

Proof. Consider an argument $\langle \Phi, \alpha \rangle$ and write $\langle \phi_1, \ldots, \phi_n \rangle$ for the canonical enumeration of Φ. Applying Theorem 4.9, it is enough to prove the result for any undercut $\langle \Psi, \neg(\gamma_1 \wedge \cdots \wedge \gamma_m) \rangle$ of $\langle \Phi, \alpha \rangle$. By Definition 4.3, $\{\gamma_1, \ldots, \gamma_m\} \subseteq \{\phi_1, \ldots, \phi_n\}$. So, $\neg(\gamma_1 \wedge \cdots \wedge \gamma_m) \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Then, $\Psi \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ because $\langle \Psi, \neg(\gamma_1 \wedge \cdots \wedge \gamma_m) \rangle$ is an undercut, hence an argument. Accordingly, there exists a minimal subset $\Psi' \subseteq \Psi \subseteq \Delta$ such that $\Psi' \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Moreover, Ψ' is clearly consistent. Therefore, $\langle \Psi', \neg(\phi_1 \wedge \cdots \wedge \phi_n) \rangle$ is an argument. Obviously, it is an undercut for $\langle \Phi, \alpha \rangle$. It is a canonical undercut for $\langle \Phi, \alpha \rangle$ in view of Theorem 5.4. It is more conservative than $\langle \Psi, \neg(\gamma_1 \wedge \cdots \wedge \gamma_m) \rangle$ as we have $\Psi' \subseteq \Psi$ and $\neg(\gamma_1 \wedge \cdots \wedge \gamma_m) \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. □

6. Argument trees

An argument tree describes the various ways an argument can be challenged, as well as how the counter-arguments to the initial argument can themselves be challenged, and so on recursively.

Definition 6.1. An *argument tree* for α is a tree where the nodes are arguments such that

(1) The root is an argument for α.
(2) For no node $\langle \Phi, \beta \rangle$ with ancestor nodes $\langle \Phi_1, \beta_1 \rangle, \ldots, \langle \Phi_n, \beta_n \rangle$ is Φ a subset of $\Phi_1 \cup \cdots \cup \Phi_n$.
(3) The children nodes of a node N consist of all canonical undercuts for N that obey (2).

We first give an illustration of an argument tree in Example 6.2 and then we motivate the conditions of Definition 6.1 as follows: condition (2) is meant to avoid the situation illustrated by Example 6.3; and condition (3) is meant to avoid the situation illustrated by Example 6.4.

Example 6.2. Given $\Delta = \{\alpha, \alpha \to \beta, \gamma, \gamma \to \neg\alpha, \neg\gamma \vee \neg\alpha\}$, we have the following argument tree.

$$\begin{array}{ccc} & \langle\{\alpha, \alpha \to \beta\}, \beta\rangle & \\ \nearrow & & \nwarrow \\ \langle\{\gamma, \gamma \to \neg\alpha\}, \neg(\alpha \wedge (\alpha \to \beta))\rangle & & \langle\{\gamma, \neg\gamma \vee \neg\alpha\}, \neg(\alpha \wedge (\alpha \to \beta))\rangle \end{array}$$

Note the two undercuts are equivalent. They do count as two arguments because they are based on two different items of the database (even though these items turn out to be logically equivalent).

We adopt a lighter notation, writing $\langle\Psi, \diamond\rangle$ for a canonical undercut of $\langle\Phi, \beta\rangle$. Clearly, $\diamond$ is $\neg(\phi_1 \wedge \cdots \wedge \phi_n)$ where $\langle\phi_1, \ldots, \phi_n\rangle$ is the canonical enumeration for Φ.

Example 6.3. Let $\Delta = \{\alpha, \alpha \to \beta, \gamma \to \neg\alpha, \gamma\}$.

$$\begin{array}{c} \langle\{\alpha, \alpha \to \beta\}, \beta\rangle \\ \uparrow \\ \langle\{\gamma, \gamma \to \neg\alpha\}, \diamond\rangle \\ \uparrow \\ \langle\{\alpha, \gamma \to \neg\alpha\}, \diamond\rangle \end{array}$$

This is not an argument tree because the undercut to the undercut is actually making exactly the same point (that α and γ are incompatible) as the undercut itself does, just by using modus tollens instead of modus ponens.

Example 6.4. Given $\Delta = \{\alpha, \beta, \alpha \to \gamma, \beta \to \delta, \neg\alpha \vee \neg\beta\}$, consider the following tree.

$$\begin{array}{ccc} & \langle\{\alpha, \beta, \alpha \to \gamma, \beta \to \delta\}, \gamma \wedge \delta\rangle & \\ \nearrow & & \nwarrow \\ \langle\{\alpha, \neg\alpha \vee \neg\beta\}, \neg\beta\rangle & & \langle\{\beta, \neg\alpha \vee \neg\beta\}, \neg\alpha\rangle \end{array}$$

This is not an argument tree because the two children nodes are not maximally conservative undercuts. The first undercut is essentially the same argument as the second undercut in a rearranged form (relying on α and β being incompatible, assume one and then conclude that the other doesn't hold). If we replace these by the maximally conservative undercut $\langle\{\neg\alpha \vee \neg\beta\}, \diamond\rangle$, we obtain an argument tree.

The following result is important in practice—particularly in light of Corollary 4.12 and also other results we present in the next section.

Theorem 6.5. *Argument trees are finite.*

Proof. Since Δ is finite, there are only a finite number of subsets of Δ. By condition (2) of Definition 6.1, no branch in an argument tree can then be infinite. Also, there is then a finite number of canonical undercuts (Definition 5.3). By condition (3) of Definition 6.1, the branching factor in an argument tree is finite. □

Theorem 6.6. *If Δ is consistent, then all argument trees have exactly one node. The converse is true when no formula in Δ is inconsistent.*

Proof. Apply Corollary 4.19. □

The form of an argument tree is not arbitrary. It summarizes all lines of discussion about the argument in the root node. Each node except the root node is the starting point of an implicit series of related arguments. In the next section, we look more closely at the nature of these related arguments.

7. Duplicates

Equivalent arguments are arguments that express the same reason for the same point. For undercuts, a more refined notion than equivalent arguments is useful:

Definition 7.1. Two undercuts $\langle \Gamma \cup \Phi, \neg\psi \rangle$ and $\langle \Gamma \cup \Psi, \neg\phi \rangle$ are *duplicates* of each other iff ϕ is $\phi_1 \wedge \cdots \wedge \phi_n$ such that $\Phi = \{\phi_1, \ldots, \phi_n\}$ and ψ is $\psi_1 \wedge \cdots \wedge \psi_m$ such that $\Psi = \{\psi_1, \ldots, \psi_m\}$.

Duplicates introduce a symmetric relation which fails to be transitive (and reflexive, as it is actually antireflexive). Arguments which are duplicates of each other are essentially the same argument in a rearranged form.

Example 7.2. The two arguments below are duplicates of each other.

$$\langle \{\alpha, \neg\alpha \vee \neg\beta\}, \neg\beta \rangle$$
$$\langle \{\beta, \neg\alpha \vee \neg\beta\}, \neg\alpha \rangle$$

Example 7.3. To illustrate the lack of transitivity in the duplicate relationship, the following two arguments are duplicates,

$$\langle \{\gamma, \alpha, \alpha \wedge \gamma \to \neg\beta\}, \neg\beta \rangle$$
$$\langle \{\gamma, \beta, \alpha \wedge \gamma \to \neg\beta\}, \neg\alpha \rangle$$

and similarly the following two arguments are duplicates,

$$\langle \{\gamma, \beta, \alpha \wedge \gamma \to \neg\beta\}, \neg\alpha \rangle$$
$$\langle \{\alpha, \alpha \wedge \gamma \to \neg\beta\}, \neg(\beta \wedge \gamma) \rangle$$

but the following two are not duplicates.

$$\langle\{\alpha, \alpha \wedge \gamma \rightarrow \neg\beta\}, \neg(\beta \wedge \gamma)\rangle$$
$$\langle\{\gamma, \alpha, \alpha \wedge \gamma \rightarrow \neg\beta\}, \neg\beta\rangle$$

The following theorem shows how we can systematically obtain duplicates. In this result, we see there is an explosion of duplicates for each maximally conservative undercut. This obviously is a potential concern for collating counter-arguments.

Theorem 7.4. *For every maximally conservative undercut $\langle\Psi, \beta\rangle$ to an argument $\langle\Phi, \alpha\rangle$ such that β is logically equivalent with Ψ, there exist at least $2^m - 1$ arguments each of which undercuts the undercut (m is the size of Ψ). Each of these $2^m - 1$ arguments is a duplicate to the undercut.*

Proof. By Theorem 5.2, β is $\neg(\phi_1 \wedge \cdots \wedge \phi_n)$ where $\Phi = \{\phi_1, \ldots, \phi_n\}$. Let $\Psi = \{\psi_1, \ldots, \psi_m\}$. According to the assumption, $\Psi \nvdash \bot$ and Ψ is a minimal subset of Δ such that

$$\Psi \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n).$$

We show the result for each non-empty subset of Ψ, which gives us $2^m - 1$ cases, considering only $\{\psi_1, \ldots, \psi_p\} \subseteq \{\psi_1, \ldots, \psi_m\}$ in the proof as the case is clearly the same for all sets $\{\psi_1, \ldots, \psi_p\}$ that can be selected from all possible permutations of $\langle\psi_1, \ldots, \psi_m\rangle$: We show that $\langle\{\phi_1, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\}, \neg(\psi_1 \wedge \cdots \wedge \psi_p)\rangle$ is an argument.

By the hypothesis $\{\psi_1, \ldots, \psi_m\} \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$, we get $\{\phi_1, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\} \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_p)$ according to classical logic.

If we assume $\{\phi_1, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\} \vdash \bot$, we would get $\{\psi_{p+1}, \ldots, \psi_m\} \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Therefore, $\langle\{\psi_1, \ldots, \psi_m\}, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ would not be an argument because $\{\psi_1, \ldots, \psi_m\}$ would not be a minimal subset of Δ entailing $\neg(\phi_1 \wedge \cdots \wedge \phi_n)$. This would contradict the fact that $\langle\Psi, \neg(\phi_1 \wedge \cdots \wedge \phi_n)\rangle$ is an undercut, hence an argument. That is, we have proven that $\{\phi_1, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\} \nvdash \bot$.

There only remains to show that $\{\phi_1, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\}$ is a minimal subset of Δ for $\neg(\psi_1 \wedge \cdots \wedge \psi_p)$ to be entailed. We use reductio ad absurdum, assuming that at least one ϕ_i or ψ_j is unnecessary when deducing $\neg(\psi_1 \wedge \cdots \wedge \psi_p)$. In symbols, either $\{\phi_2, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\} \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_p)$ or $\{\phi_1, \ldots, \phi_n, \psi_{p+2}, \ldots, \psi_m\} \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_p)$ (again, all cases are symmetric so it is enough to consider only $i = 1$ and $j = p + 1$).

Let us first assume $\{\phi_1, \ldots, \phi_n, \psi_{p+2}, \ldots, \psi_m\} \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_p)$. As a consequence, $\{\psi_1, \ldots, \psi_p, \psi_{p+2}, \ldots, \psi_m\} \vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. This would contradict $\{\psi_1, \ldots, \psi_m\}$ being a minimal subset of Δ entailing $\neg(\phi_1 \wedge \cdots \wedge \phi_n)$.

Turning to the second case, let us assume that $\{\phi_2, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\} \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_p)$. Thus, $\{\psi_1, \ldots, \psi_m\} \vdash \neg(\phi_2 \wedge \cdots \wedge \phi_n)$. Hence $\neg(\phi_1 \wedge \cdots \wedge \phi_n) \vdash \neg(\phi_2 \wedge \cdots \wedge \phi_n)$ because β is logically equivalent with Ψ by the condition in the theorem. It follows that $\phi_2 \wedge \cdots \wedge \phi_n \vdash \phi_1 \wedge \cdots \wedge \phi_n$. As a consequence, $\Phi = \{\phi_1, \ldots, \phi_n\}$ cannot be a minimal subset of Δ entailing α and this contradicts the definition of an argument as applied to $\langle \Phi, \alpha \rangle$.

Either case yields a contradiction, hence no proper subset of $\{\phi_1, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\}$ entails $\neg(\psi_1 \wedge \cdots \wedge \psi_p)$. So, $\langle \{\phi_1, \ldots, \phi_n, \psi_{p+1}, \ldots, \psi_m\}, \neg(\psi_1 \wedge \cdots \wedge \psi_p) \rangle$ is an argument and it clearly is an undercut to the undercut.

Verification that it is a duplicate of the undercut is routine. □

Theorem 7.5. *No two maximally conservative undercuts of the same argument are duplicates.*

Proof. Let $\langle \Gamma \cup \Theta, \neg(\alpha_1 \wedge \cdots \wedge \alpha_n) \rangle$ and $\langle \Gamma \cup \Theta', \neg(\alpha'_1 \wedge \cdots \wedge \alpha'_m) \rangle$ be two maximally conservative undercuts for $\langle \Phi, \beta \rangle$ that are duplicates of each other. Then, Θ is logically equivalent with $\alpha'_1 \wedge \cdots \wedge \alpha'_m$. Hence, $\Gamma \cup \Theta \vdash \alpha'_1 \wedge \cdots \wedge \alpha'_m$ while $\Gamma \cup \Theta \vdash \neg(\alpha_1 \wedge \cdots \wedge \alpha_n)$. According to Theorem 5.2, $\Phi = \{\alpha_1, \ldots, \alpha_n\} = \{\alpha'_1, \ldots, \alpha'_m\}$. That is, $\alpha_1 \wedge \cdots \wedge \alpha_n$ and $\alpha'_1 \wedge \cdots \wedge \alpha'_m$ are logically equivalent. Therefore, $\Gamma \cup \Theta \vdash \bot$. This would contradict the fact that $\Gamma \cup \Theta$ is the support of an argument. □

Theorem 7.6. *No branch in an argument tree contain duplicates, except possibly for the child of the root to be a duplicate to the root.*

Proof. Assume the contrary: There is $\langle \Gamma \cup \Phi, \neg\psi \rangle$ which is an ancestor node of $\langle \Gamma \cup \Psi, \neg\phi \rangle$ where Φ is $\{\phi_1, \ldots, \phi_n\}$, Ψ is $\{\psi_1, \ldots, \psi_m\}$, ϕ is $\phi_1 \wedge \cdots \wedge \phi_n$ and ψ is $\psi_1 \wedge \cdots \wedge \psi_m$. By definition of an argument tree, $\langle \Gamma \cup \Psi, \neg\phi \rangle$ is a canonical undercut for its parent node, which is $\langle \Phi, \alpha \rangle$ for some α and which also has $\langle \Gamma \cup \Phi, \neg\psi \rangle$ as an ancestor node. Unless $\langle \Phi, \alpha \rangle$ is the same node as $\langle \Gamma \cup \Phi, \neg\psi \rangle$, this contradicts condition (2) of Definition 6.1 because Φ is of course a subset of $\Gamma \cup \Phi$. In case $\langle \Phi, \alpha \rangle$ coincides with $\langle \Gamma \cup \Phi, \neg\psi \rangle$, then $\Gamma \subseteq \Phi$. Also, $\langle \Phi, \neg\psi \rangle$ is the parent node for $\langle \Gamma \cup \Psi, \neg\phi \rangle$. Should the former not be the root, it has a parent node of the form $\langle \Psi, \beta \rangle$ for some β. Therefore, $\langle \Psi, \beta \rangle$ is an ancestor node of $\langle \Gamma \cup \Psi, \neg\phi \rangle$. Moreover, $\langle \Phi, \neg\psi \rangle$ has already been proven the parent node for $\langle \Gamma \cup \Psi, \neg\phi \rangle$ where $\Gamma \subseteq \Phi$. So, condition (2) of Definition 6.1 is again contradicted.[2] □

These last two results are important. They show that argument trees are an efficient and lucid way of representing the pertinent counter-arguments to each argument: Theorem 7.5 shows it regarding breadth and Theorem 7.6 shows it regarding depth. Moreover, they show that the intuitive need to eliminate duplicates from argument trees is taken care of through an efficient syntactical criterion (condition (2) of Definition 6.1).

[2] Strictly speaking, it is assumed throughout the proof that rewriting (a canonical enumeration of) a subset of Δ into a conjunctive formula is an injective function. Such a restriction is inessential because it only takes an easy procedure to have the assumption guaranteed to hold.

8. Argument structures

We now consider how we can gather argument trees for and against a point. To do this, we define argument structures.

Definition 8.1. An *argument structure* for a formula α is a pair of sets $\langle \mathcal{P}, \mathcal{C} \rangle$ where $\mathcal{P}$ is the set of argument trees for α and $\mathcal{C}$ is the set of argument trees for $\neg\alpha$.

Example 8.2. Let $\Delta = \{\alpha \vee \beta, \alpha \rightarrow \gamma, \neg\gamma, \neg\beta, \delta \leftrightarrow \beta\}$. For this, we obtain three argument trees for the argument structure for $\alpha \vee \neg\delta$.

$$\mathcal{P} \left\{ \begin{array}{cc} \langle \{\alpha \vee \beta, \neg\beta\}, \alpha \vee \neg\delta \rangle & \langle \{\delta \leftrightarrow \beta, \neg\beta\}, \alpha \vee \neg\delta \rangle \\ \uparrow & \uparrow \\ \langle \{\alpha \rightarrow \gamma, \neg\gamma\}, \Diamond \rangle & \langle \{\alpha \vee \beta, \alpha \rightarrow \gamma, \neg\gamma\}, \Diamond \rangle \end{array} \right.$$

$$\mathcal{C} \left\{ \begin{array}{c} \langle \{\alpha \vee \beta, \alpha \rightarrow \gamma, \neg\gamma, \delta \leftrightarrow \beta\}, \neg(\alpha \vee \neg\delta) \rangle \\ \uparrow \\ \langle \{\neg\beta\}, \Diamond \rangle \end{array} \right.$$

Example 8.3. Let $\Delta = \{\alpha \leftrightarrow \beta, \beta \vee \gamma, \gamma \rightarrow \beta, \neg\alpha \vee \neg\beta \vee \neg\gamma, \gamma \wedge \delta, \neg\delta\}$. From this we obtain the following argument trees for and against $\beta \wedge \delta$.

$$\begin{array}{ccc} & \langle \{\gamma \wedge \delta, \gamma \rightarrow \beta\}, \beta \wedge \delta \rangle & \\ \nearrow & \uparrow & \nwarrow \\ \langle \{\alpha \leftrightarrow \beta, \neg\alpha \vee \neg\beta \vee \neg\gamma, \gamma \wedge \delta\}, \diamond \rangle & \langle \{\neg\delta\}, \diamond \rangle & \langle \Delta \setminus \{\gamma \wedge \delta, \neg\delta\}, \diamond \rangle \\ \uparrow & & \\ \langle \{\neg\delta\}, \diamond \rangle & & \end{array}$$

$$\begin{array}{ccc} & \langle \{\alpha \leftrightarrow \beta, \neg\alpha \vee \neg\beta \vee \neg\gamma, \gamma \wedge \delta\}, \neg(\beta \wedge \delta) \rangle & \\ \nearrow & & \nwarrow \\ \langle \{\gamma \rightarrow \beta\}, \diamond \rangle & & \langle \{\neg\delta\}, \diamond \rangle \end{array}$$

$$\begin{array}{c} \langle \{\neg\delta\}, \neg(\beta \wedge \delta) \rangle \\ \uparrow \\ \langle \{\gamma \wedge \delta\}, \diamond \rangle \end{array}$$

Theorem 8.4. *Let $\langle \mathcal{P}, \mathcal{C} \rangle$ be an argument structure. If there exists an argument tree in $\mathcal{P}$ that has exactly one node, then $\mathcal{C}$ is the empty set. The converse is untrue, even when assuming that $\mathcal{P}$ is non-empty.*

Proof. Assume the contrary: $\mathcal{P}$ contains an argument tree consisting of a single argument while $\mathcal{C}$ is non-empty. Let the single node tree in $\mathcal{P}$ be the argument $\langle \Phi, \alpha \rangle$ and let the root node of the tree in $\mathcal{C}$ be the argument $\langle \Psi, \neg\alpha \rangle$. By the definition of a rebuttal and Theorem 4.6, $\langle \Psi, \neg\alpha \rangle$ is a defeater for $\langle \Phi, \alpha \rangle$. According to Theorem 5.8, there then exists a canonical undercut for $\langle \Phi, \alpha \rangle$ and this contradicts the fact that there is an argument tree consisting only of $\langle \Phi, \alpha \rangle$. $\square$

Example 8.5. Let $\Delta = \{\alpha \vee \neg\beta, \beta, \neg\beta\}$. In the argument structure $\langle \mathcal{P}, \mathcal{C} \rangle$ for α, we have that $\mathcal{C}$ is the empty set while $\mathcal{P}$ contains an argument tree which has more than one node:

$$\begin{array}{c} \langle \{\alpha \vee \neg\beta, \beta\}, \alpha \rangle \\ \uparrow \\ \langle \{\neg\beta\}, \diamond \rangle \end{array}$$

Example 8.5 illustrates the last sentence in Theorem 8.4. If Δ is augmented with $\alpha \wedge \gamma$ for instance, then $\langle \mathcal{P}, \mathcal{C} \rangle$ is such that $\mathcal{P}$ contains both an argument tree with more than one node and an argument tree consisting of just a root node.

Theorem 8.6. *Let $\langle \mathcal{P}, \mathcal{C} \rangle$ be an argument structure such that $\mathcal{P}$ is non-empty. If Δ is consistent, then each argument tree in $\mathcal{P}$ has exactly one node and $\mathcal{C}$ is the empty set. The converse is untrue, even when assuming that each formula in Δ is consistent.*

Proof. For each argument which is a root node in an argument tree of $\mathcal{P}$, apply Corollary 4.19. So, we have just proved that each member of $\mathcal{P}$ consists of a single argument. As for $\mathcal{C}$ being the empty set, it is then enough to apply Theorem 8.4. $\square$

The last sentence in the statement of Theorem 8.6 can be illustrated by the following counter-example.

Example 8.7. Let $\Delta = \{\alpha, \beta, \neg\beta\}$. The argument structure $\langle \mathcal{P}, \mathcal{C} \rangle$ for α is such that $\mathcal{P}$ contains a single argument tree consisting of just the root node below:

$$\langle \{\alpha\}, \alpha \rangle$$

In argument structures, $\mathcal{P}$ and $\mathcal{C}$ are symmetrical. Any property enjoyed by one has a counterpart, which is a property enjoyed by the other: Both are the same property, with $\mathcal{P}$ and $\mathcal{C}$ exchanged. E.g., we have the result similar to Theorem 8.4 stating that if there exists an argument tree in $\mathcal{C}$ which has exactly one node, then $\mathcal{P}$ is the empty set. Symmetry goes even deeper, inside the argument trees of $\mathcal{P}$ and $\mathcal{C}$. This is exemplified in the next result.

Theorem 8.8. *Let $\langle [X_1, \ldots, X_n], [Y_1, \ldots, Y_m] \rangle$ be an argument structure. For any i and any j, the support of the root node of Y_j (respectively X_i) is a superset of the support of a canonical undercut for the root node of X_i (respectively Y_j).*

Proof. Let $\langle \Phi, \alpha \rangle$ be the root node of X_i and $\langle \Psi, \neg\alpha \rangle$ be the root node of Y_j. Then, $\Phi \vdash \alpha$ and $\Psi \vdash \neg\alpha$. So, $\Phi \cup \Psi \vdash \bot$. Accordingly, $\Psi \vdash \neg\phi$ where ϕ is $\phi_1 \wedge \cdots \wedge \phi_k$ and $\langle \phi_1, \ldots, \phi_k \rangle$ is the canonical enumeration of Φ. Also, Ψ is consistent because $\langle \Psi, \neg\alpha \rangle$ is an argument. Let Ψ' be a minimal subset of Ψ such that $\Psi' \vdash \neg\phi$. Then, $\langle \Psi', \neg\phi \rangle$ is an argument. It clearly is a canonical undercut for $\langle \Phi, \alpha \rangle$. $\square$

Theorem 8.8 is reminiscent of the phenomenon reported in Corollary 4.10.

Theorem 8.9. *Let $\langle \mathcal{P}, \mathcal{C} \rangle$ be an argument structure. Then, both $\mathcal{P}$ and $\mathcal{C}$ are finite.*

Proof. We only prove the result for $\mathcal{P}$. Clearly, no two argument trees have the same root node. Therefore, all argument trees of $\mathcal{P}$ have a different support. So, there can only be as many argument trees of $\mathcal{P}$ as there are subsets of Δ. These are finitely many because Δ is finite. $\square$

Definition 8.10. A *categoriser* is a mapping from argument trees to numbers. A categorisation is then a pair of multisets obtained by applying the same categorizer to each argument tree in an argument structure.

The number assigned by a categoriser is intended to capture the relative strength of an argument taking into account the undercuts, undercuts to undercuts, and so on. In other words, it is an attempt to provide an abstraction of an argument tree in the form of a single number.

The *h-categoriser*, denoted h, is an example of a categoriser. An argument tree of root R is assigned a number $h(R)$ defined recursively by

$$h(N) = \frac{1}{1 + h(N_1) + \cdots + h(N_l)},$$

where $N_1, \ldots, N_l$ are the children nodes for N (if $l = 0$, $h(N_1) + \cdots + h(N_l) = 0$).

The intuitive idea about the h-categoriser is that the value of an argument is maximum if it has no undercuts because the "decrease" is minimum: The more undercuts an argument has, the less its value is. Recursively, the value of the argument is minimum if no undercut has itself an undercut because the "decrease of the decrease" is maximum: The more undercuts there are to the undercuts of an argument, the more its value is (all other things being equal, e.g., the number of undercuts to the argument is fixed).

Definition 8.11. An *accumulator* is a function that takes a categorisation for a formula α and returns a pair of numbers (α^+, α^-) where α^+ is the accumulated value for α, and α^- is the accumulated value against α. The balance of the accumulated values is calculated as $\alpha^+ - \alpha^-$.

So, if the balance of accumulated values is 0, then the arguments for the formula "equal" the arguments against the formula. If the balance of accumulated values is positive, then the arguments for the formula are stronger (when aggregated) than the arguments against the formula, and if the balance of accumulated values is negative, then the arguments for the formula are weaker (when aggregated) than the arguments against the formula.

The *log-accumulator* is an example of an accumulator. For a categorisation $\langle X, Y\rangle$ where $X = [X_1, \ldots, X_n]$ and $Y = [Y_1, \ldots, Y_m]$, define

$$l(\langle X, Y\rangle) = \bigl(\log(1 + X_1 + \cdots + X_n), \log(1 + Y_1 + \cdots + Y_m)\bigr).$$

The idea of the log-accumulator is that the added value contributed by an argument to the overall value of a set of similar arguments (i.e., which agree with it) is more important when the set is smaller: The overall value of a couple of similar arguments is intuitively more than $2/75$ of the value of a bunch of 75 similar arguments, which in turn is intuitively much more than $75/100$ of the value of a series of 100 similar arguments. More generally, its makes almost as little difference whether one has hundreds of similar arguments or thousands while, by contrast, it does make a significant difference whether one has three similar arguments or only two, and the difference is even bigger whether one has two similar arguments or just a single argument.

We do not ascribe any normative or prescriptive dimension to the h-categorizer or the log-accumulator. We simply want to show that our approach is versatile enough to cope with various ideas about aggregation, including the possibility to take into account the number of arguments. Here are two examples.

Example 8.12. Consider the categorisation $\langle[1], [1/2]\rangle$. Using the log-accumulator function, we get 0.47 as the balance of the accumulated values. Now consider the categorisation $\langle[1, 1/2], [1/2, 1/2]\rangle$. Using the log-accumulator function, we obtain 0.41 as the balance of the accumulated values. So we can see that adding an argument tree of value $1/2$ to both the pro and con sides benefits the con side since initially the con side is a much weaker argument than the pro side.

Example 8.13. For the categorisation $\langle[1/2, 1/2], [1]\rangle$, the log-accumulator function gives -0.25 as the balance of the accumulated values. Now consider the categorisation $\langle[1/2, 1/2, 1/2], [1, 1/2]\rangle$. Using the log-accumulator function, we obtain -0.29 as the balance of the accumulated values. So we can see that adding the argument trees of value $1/2$ to both the pro and con sides benefits the con side since initially the pro side has two arguments of value $1/2$ but the con side has a single argument of value 1 (in particular, we want an argument to have a more profound effect when confirming a single argument than when joining a hundred similar arguments which mutually agree).

Through Definition 8.10 and Definition 8.11, there are many possible categoriser and accumulator functions that could be developed for applications. Furthermore, it may be possible to develop particular categoriser and accumulator functions that conform to either probabilistic or possibilistic approaches.

We give other examples of categoriser and accumulator functions in the next section.

9. Comparison with other frameworks

In this section, we compare our framework with other approaches to argument aggregation. This comparison is conducted largely by showing how a diverse range of other approaches can be directly incorporated within our framework. To do this, we provide definitions for categoriser and accumulator functions for each of these other approaches.

However, since a number of approaches use a weaker notion of an argument than in our framework, we need to adapt our definition for an argument. In particular, a number of argumentation systems use a restricted language—for example a language composed of literals plus rules of the following form where $\alpha_1, \ldots, \alpha_n, \beta$ are literals:

$$\alpha_1 \wedge \cdots \wedge \alpha_n \to \beta$$

The proof theory for generating arguments in these systems is then just a form of generalised modus ponens. To show these systems are a special case of our system, we adopt revised definitions for the language and arguments. Essentially, the languages allowed are sub-languages of classical logic, and the arguments allowed are those derived by an appropriate sub-theory of classical proof theory.

9.1. *Binary argumentation*

Many of the definitions for argumentation are based on an approach that we describe as a form of binary argumentation. For examples of logics that we view as a form of binary argumentation, see [10,14,17,18,22,26]. This section is intended to show that this "simple" form of argumentation can be captured in our framework.

Before giving the appropriate definitions for the categoriser and accumulator functions, we need to introduce a few notions.

If A, B and C are three arguments such that A is undercut by B and B is undercut by C then C is called a defence for A. We define the "defend" relation as the transitive closure of "being a defence".

An argument tree is said to be successful iff every leaf defends the root node.

Definition 9.1. The *binary categoriser* is a function, denoted s, from the set of argument trees to $\{0, 1\}$ such that $s(T) = 1$ iff T is successful.

Definition 9.2. The *binary accumulator* is a function, denoted b_a, from categorisations to the set $\{(1,1), (1,0), (0,1), (0,0)\}$. Let $\langle X, Y\rangle$ be a categorisation, then

$$b_a(\langle X, Y\rangle) = \big(w(X), w(Y)\big)$$

where $w(Z) = 1$ iff $1 \in Z$.

The results for the binary accumulator can be given the following interpretation:

$(1, 1)$ means the arguments for α and $\neg\alpha$ mutually rebut.

$(1, 0)$ means α follows.

$(0, 1)$ means $\neg\alpha$ follows.

$(0, 0)$ means there are no arguments for α or $\neg\alpha$ that prevail.

Example 9.3. Consider $\Delta = \{\alpha \leftrightarrow \neg\delta, \beta, \beta \to \alpha, \gamma \wedge \neg\beta, \neg\gamma, \delta, \neg\delta\}$. We obtain the following argument trees for and against α.

$$
\begin{array}{lccccc}
 & & & \langle\{\beta \to \alpha, \beta\}, \alpha\rangle & & \\
 & & \nearrow & & \nwarrow & \\
T_1 & \langle\{\gamma \wedge \neg\beta\}, \diamond\rangle & & & & \langle\{\delta, \alpha \leftrightarrow \neg\delta\}, \diamond\rangle \\
 & \uparrow & & & & \uparrow \\
 & \langle\{\neg\gamma\}, \diamond\rangle & & & & \langle\{\neg\delta\}, \diamond\rangle
\end{array}
$$

$$
\begin{array}{lc}
 & \langle\{\neg\delta, \alpha \leftrightarrow \neg\delta\}, \alpha\rangle \\
T_2 & \uparrow \\
 & \langle\{\delta\}, \diamond\rangle
\end{array}
$$

$$
\begin{array}{lccccc}
 & & & \langle\{\delta, \alpha \leftrightarrow \neg\delta\}, \neg\alpha\rangle & & \\
T_3 & & \nearrow & & \nwarrow & \\
 & \langle\{\beta, \beta \to \alpha\}, \diamond\rangle & & & & \langle\{\neg\delta\}, \diamond\rangle \\
 & \uparrow & & & & \\
 & \langle\{\gamma \wedge \neg\beta\}, \diamond\rangle & & & & \\
 & \uparrow & & & & \\
 & \langle\{\neg\gamma\}, \diamond\rangle & & & &
\end{array}
$$

From these trees, applying the binary categorizer gives the following values: $s(T_1) = 1$, $s(T_2) = 0$, and $s(T_3) = 0$. Applying the binary accumulator gives $b_a(\langle[1, 0], [0]\rangle) = (w([1, 0]), w([0])) = (1, 0)$. Therefore, α follows from Δ using binary argumentation.

Since the language used for binary argumentation systems is weaker than classical logic, we will not provide a formal comparison with our framework here.

9.2. *Counting arguments for and against*

Whilst most proposals for argument aggregation incorporate some form of binary aggregation function, there are some proposals for non-binary aggregation functions that are based on counting the number of arguments for and against a conclusion, and if there are more arguments for the conclusion, then the conclusion follows, otherwise it is defeated (see for example [3]). We formalise this approach as follows:

Definition 9.4. The *counting categoriser*, denoted c, is an example of categoriser where the co-domain is $\{1\}$, and so for any argument tree T, $c(T) = 1$.

Definition 9.5. The *counting accumulator*, denoted c_a, is an example of an accumulator. Let $\langle [X_1, \ldots, X_n], [Y_1, \ldots, Y_m] \rangle$ be a categorisation, then

$$c_a(\langle [X_1, \ldots, X_n], [Y_1, \ldots, Y_m] \rangle) = (n, m).$$

Example 9.6. Let $\Delta = \{\beta \to \alpha, \delta \to \alpha, \gamma \to \neg\alpha, \beta, \gamma, \delta\}$. From this, we have two arguments for α and one against, and so the result of applying the counting accumulator is $(2, 1)$.

Since the language used for argumentation systems based on counting is weaker than classical logic, we will not provide a formal comparison with our framework here.

9.3. Comparison with argumentative logics

Argumentative logics [8] overlap with a number of other approaches to reasoning with maximal consistent subsets of data including [2,21]. We summarize the inferencing available with some of the key argumentative logics as follows.

Definition 9.7. Let α be a classical formula and let Δ be a set of classical formulae.

- α is an existential inference from Δ iff there is a maximal consistent subset of Δ where α is a classical deduction.
- α is unrebutted inference from Δ iff there is a maximal consistent subset of Δ where α is a classical deduction and there is no maximal consistent subset of Δ where $\neg\alpha$ is a classical deduction.
- α is a universal inference from Δ iff for all maximal consistent subsets of Δ, α is a classical deduction.
- α is a free inference from Δ iff for the intersection of all maximal consistent subsets of Δ, α is a classical deduction.

In the following subsections, we consider each of these approaches to reasoning with respect to our framework. For this we require the following two definitions.

Definition 9.8. The *unit categoriser* is a function, denoted c, from the set of argument trees to $\{1\}$ such that $c(T) = 1$ in all cases.

Definition 9.9. The *unit accumulator* is a function, denoted u_a, from the set of categorisations to the set $\{(1, 1), (1, 0), (0, 1), (0, 0)\}$ such that for a categorisation $\langle X, Y \rangle$

$$u_a(\langle X, Y \rangle) = \big(p(X), p(Y)\big),$$

where $p(Z) = 1$ iff $Z \neq \emptyset$.

9.3.1. Existential inferencing

Theorem 9.10. *If applying the unit categoriser to the argument structure for α yields $\langle X, Y\rangle$, then α is an existential inference from Δ iff $u_a(\langle X, Y\rangle) = (1, n)$ for some $n \in \{0, 1\}$.*

Proof. Let $\langle \mathcal{P}, \mathcal{C}\rangle$ be the argument structure for α. Clearly, $\mathcal{P}$ contains an argument tree for α iff there exists a (maximal) consistent subset of Δ entailing α. By Definition 9.8, the unit categorisation of $\langle \mathcal{P}, \mathcal{C}\rangle$ is $\langle X, Y\rangle$ where $X \neq \emptyset$ iff $\mathcal{P} \neq \emptyset$. By Definition 9.9, $u_a(\langle X, Y\rangle) = (1, n)$ iff $X \neq \emptyset$. So, α is an existential inference from Δ iff $u_a(\langle X, Y\rangle) = (1, n)$ (the value of n is 0 or 1 depending on whether Y is empty or not). □

9.3.2. Unrebutted inferencing

Theorem 9.11. *α is an unrebutted inference from Δ iff $u_a(\langle X, Y\rangle) = (1, 0)$ where $\langle X, Y\rangle$ results from applying the unit categorizer to the argument structure for α.*

Proof. Like the proof of Theorem 9.10 for $Y = \emptyset$, that is, $\mathcal{C} = \emptyset$ (behaving like $\mathcal{P} = \emptyset$ above). □

9.3.3. Universal inferencing

Universal inferencing is difficult to capture in our framework. In universal inferencing we need to cross-check the consistency of arguments in the different trees. Consider the following two examples of argument structures for α. The argument structures are isomorphic, but in the first example α is not a universal inference, whereas it is a universal inference in the second example.

Example 9.12. Let $\Delta = \{\beta, \beta \to \alpha, \neg\beta, \gamma, \gamma \to \alpha, \neg\gamma\}$. Here there are two argument trees for α.

$$\begin{array}{cc} \langle\{\beta, \beta \to \alpha\}, \alpha\rangle & \langle\{\gamma, \gamma \to \alpha\}, \alpha\rangle \\ \uparrow & \uparrow \\ \langle\{\neg\beta\}, \Diamond\rangle & \langle\{\neg\gamma\}, \Diamond\rangle \end{array}$$

For Δ there are four maximal consistent subsets:

S_1 $\{\beta, \beta \to \alpha, \gamma, \gamma \to \alpha, \} \vdash \alpha$

S_2 $\{\beta, \beta \to \alpha, \gamma \to \alpha, \neg\gamma\} \vdash \alpha$

S_3 $\{\beta \to \alpha, \neg\beta, \gamma, \gamma \to \alpha, \} \vdash \alpha$

S_4 $\{\beta \to \alpha, \neg\beta, \gamma \to \alpha, \neg\gamma\} \nvdash \alpha$

However, only the maximal consistent subsets S_1, S_2, and S_3 imply α. Therefore α is not a universal inference from Δ.

Example 9.13. Let $\Delta = \{\beta, \beta \rightarrow \alpha, \neg\beta, \neg\beta \rightarrow \alpha\}$. Here there are two argument trees for α.

$$\begin{array}{cc} \langle\{\beta, \beta \rightarrow \alpha\}, \alpha\rangle & \langle\{\neg\beta, \neg\beta \rightarrow \alpha\}, \alpha\rangle \\ \uparrow & \uparrow \\ \langle\{\neg\beta\}, \diamond\rangle & \langle\{\beta\}, \diamond\rangle \end{array}$$

For Δ there are two maximal consistent subsets:

S_1 $\{\beta, \beta \rightarrow \alpha, \neg\beta \rightarrow \alpha\} \vdash \alpha$

S_2 $\{\neg\beta, \neg\beta \rightarrow \alpha, \beta \rightarrow \alpha\} \vdash \alpha$

So α is a universal inference from Δ while it is not in Example 9.12, despite similar argument trees.

Since we need to know whether arguments for an inference are mutually consistent, we need to generalise the definition of a categoriser to be a function from sets of argument trees to numbers.

9.3.4. Free inferencing

Even though the usual definition of free inferencing is based on maximal consistent subsets, it can be equivalently defined without recourse to the notion of maximal consistent subsets.

Definition 9.14. The *free categoriser* is a function, denoted r, from the set of argument trees to $\{0, 1\}$ such that $r(T) = 1$ iff T is just a root node.

Definition 9.15. The *free accumulator* is a function, denoted f_a, from the set of categorisations to the set $\{(1, 1), (1, 0), (0, 1), (0, 0)\}$ such that for a categorisation $\langle X, Y\rangle$

$$f_a(\langle X, Y\rangle) = \big(w(X), w(Y)\big),$$

where $w(Z) = 1$ iff $1 \in Z$.

Theorem 9.16. *Let $\langle X, Y\rangle$ result from applying the free categoriser to the argument structure for α. Then, α is a free inference from Δ iff $f_a(\langle X, Y\rangle) = (1, 0)$.*

Proof. (Only if part) Assume that $f_a(\langle X, Y\rangle) = (1, n)$ where $\langle X, Y\rangle$ is the free categorisation obtained from $\langle\mathcal{P}, \mathcal{C}\rangle$ (the argument structure for α) and n is some value in $\{0, 1\}$. There then exists an argument tree consisting of a single node of the form $\langle\Phi, \alpha\rangle$ (cf. Definition 9.14 and Definition 9.15). This means that $\langle\Phi, \alpha\rangle$ has no canonical undercut, hence no undercut (cf. Theorem 5.8). That is, there is no $\Psi \subseteq \Delta$ such that $\Psi \nvdash \bot$ and $\Psi \vdash \neg\phi$ where ϕ is logically equivalent with Φ. Thus, there is no $\Psi \subseteq \Delta$ such that $\Psi \nvdash \bot$ and $\Psi \cup \{\phi\} \vdash \bot$. Accordingly, $\Phi \cup \Psi$ is consistent whenever Ψ is a consistent subset of Δ. By definition of a maximal consistent subset, it follows that whenever Ψ is a maximal consistent subset of Δ then $\Psi = \Phi \cup \Psi$, that is, $\Phi \subseteq \Psi$. Therefore, Φ is a subset of the intersection of all maximal consistent subsets of Δ. Since $\langle \Phi, \alpha \rangle$ is an argument, $\Phi \vdash \alpha$ which then means that α is a free inference from Δ.

(If part) Let Λ denote the intersection of all maximal consistent subsets of Δ. Assume that α is a free inference from Δ. Therefore, $\Lambda \vdash \alpha$. There exists Φ, a subset of Λ (hence, Δ) minimal with respect to entailing α (so that $\Phi \vdash \alpha$). Also, $\Phi \nvdash \perp$ (as Λ is the intersection of all maximal consistent subsets of Δ). As a consequence, $\langle \Phi, \alpha \rangle$ is an argument. Assume further that $\langle \Phi, \alpha \rangle$ has at least one undercut. That is, there exists $\Theta \subseteq \Delta$ such that $\Theta \nvdash \perp$ and $\Theta \vdash \neg\phi$ where ϕ is logically equivalent with Φ. Clearly, Θ can be extended to a maximal consistent subset of Δ, say Ω. So, $\Omega \nvdash \perp$ and $\Omega \vdash \neg\phi$. The latter amounts to $\Omega \cup \{\phi\} \vdash \perp$, or equivalently, $\Omega \cup \Phi \vdash \perp$. However, Φ is contained in the intersection of all maximal consistent subsets of Δ so that $\Omega = \Omega \cup \Phi$ holds and yields $\Omega \vdash \perp$ from which a contradiction arises. That is, there exists no undercut for $\langle \Phi, \alpha \rangle$. So, $\mathcal{P}$ contains at least the argument tree consisting of the single node $\langle \Phi, \alpha \rangle$. By Theorem 8.4, applying Definition 9.14 and Definition 9.15 then yields the result. □

9.4. Comparison with Dung's system

Here we consider Dung's system for argumentation [7].

Definition 9.17. A *Dung argumentation framework* is a pair $\Delta = \langle \Gamma, \mathcal{A} \rangle$ where Γ is a set (whose elements play the role of arguments) and $\mathcal{A} \subseteq \Gamma \times \Gamma$ (intuitively, $\mathcal{A}$ is an "attack" relation between arguments).

Superficially, an argument structure could be viewed as an argument framework in Dung's system. An argument in an argument tree could be viewed as an argument in a Dung argumentation framework, and each arc in an argument tree could be viewed as an attack relation. However, the way sets of arguments are compared is different.

Definition 9.18. A subset S of Γ is *conflict-free* if there are no two elements X, Y in S such that $X\mathcal{A}Y$ or $Y\mathcal{A}X$.

Definition 9.19. An element X in Γ is *acceptable* with respect to a subset S of Γ iff for each $Y \in \Gamma$, if $Y\mathcal{A}X$, then $Z\mathcal{A}Y$ for some $Z \in S$.

Definition 9.20. A conflict-free subset $S \subseteq \Gamma$ is *admissible* iff each element in S is acceptable with respect to S.

Definition 9.21. A *preferred extension* of an argumentation framework $\Delta = \langle \Gamma, \mathcal{A} \rangle$ is a maximal (with respect to set inclusion) admissible subset of Γ.

Some differences between Dung's approach and our approach can be seen in the following examples.

Example 9.22. Consider a set of arguments $\{a_1, a_2, a_3, a_4\}$ with the attack relation $\mathcal{A}$ such that $a_2\mathcal{A}a_1$, $a_3\mathcal{A}a_2$, $a_4\mathcal{A}a_3$, and $a_1\mathcal{A}a_4$. Here there is an admissible set $\{a_1, a_3\}$. We can try to construct an argument tree with a_1 at the root. As a counterpart to the attack relation, we regard that a_1 is undercut by a_2, a_2 is undercut by a_3, and so on. However, the corresponding sequence of nodes a_1, a_2, a_3, a_4, a_1 is not an argument tree because a_1 occurs twice in the branch (violating condition (2) of Definition 6.1). So, the form of the argument tree for a_1 fails to represent the fact that a_1 attacks a_4.

Example 9.23. Let $\Delta = \{\beta, \beta \rightarrow \alpha, \delta \wedge \neg\beta, \neg\delta \wedge \neg\beta\}$, giving the following argument tree for α.

$$\begin{array}{ccc} & \langle\{\beta, \beta \rightarrow \alpha\}, \alpha\rangle & \\ \nearrow & & \nwarrow \\ \langle\{\delta \wedge \neg\beta\}, \diamond\rangle & & \langle\{\neg\delta \wedge \neg\beta\}, \diamond\rangle \\ \uparrow & & \uparrow \\ \langle\{\neg\delta \wedge \neg\beta\}, \diamond\rangle & & \langle\{\delta \wedge \neg\beta\}, \diamond\rangle \end{array}$$

Disregarding the difference between the occurrences of $\diamond$, this argument tree rewrites as $a_2\mathcal{A}a_1$, $a_3\mathcal{A}a_1$, $a_3\mathcal{A}a_2$, and $a_2\mathcal{A}a_3$ where a_1 denotes the root node $\langle\{\beta, \beta \rightarrow \alpha\}, \alpha\rangle$. In this argument tree, each defeater of the root node is defeated. Yet no admissible set of arguments contains a_1.

Furthermore, we can show that Dung's approach is fundamentally different to our approach. We can formalise this as follows. First, for our approach, we have the following requirement:

> That an argument A attacks an argument A' means that the reasons of A contradict (in the sense of classical logic) some subset of the reasons of A'.

Clearly, the definitions for our framework that we have given in this paper meet this requirement. However, in the following result, we see that assuming this requirement in the context of Dung's framework causes a collapse. To formalise this, we rewrite the requirement as a constraint on the attack relation as follows:

$$X\mathcal{A}Y \text{ iff } R_X \cup R' \vdash \bot \text{ for some } R' \subseteq R_Y \quad \text{(Constraint 1)}$$

where R_X (respectively R_Y) is the set of reasons in X (respectively the set of reasons in Y).

Theorem 9.24. *In any instance of Dung's framework where Constraint* 1 *holds, every argument is acceptable with respect to itself, and therefore every conflict-free set of arguments is admissible.*

Proof. We show that X is acceptable with respect to $S \cup \{X\}$. So, consider $Y \in \Gamma$ such that $Y\mathcal{A}X$. That is, $R_Y \cup R'' \vdash \bot$ for some $R'' \subseteq R_X$. Then, $R_Y \cup R_X \vdash \bot$ holds. Therefore, $R_X \cup R' \vdash \bot$ for $R' = R_Y$. Hence, $X\mathcal{A}Y$ holds. □

In order to make sense, Dung's argumentation system demands the attack relation to be asymmetric although such a requirement is not stated at all in the original definitions. What is stated is only a definition of well-founded argumentation (with the outcome of having a single preferred extension) but well-founded actually means that the transitive closure of the attack relation is antisymmetric.

Of course, one could consider a variant of Dung's definition as follows: $X \in \Gamma$ is acceptable with respect to S iff for each $Y \in \Gamma$, if $Y\mathcal{A}X$ then $Z\mathcal{A}Y$ for some $Z \in S$ where $Z \neq X$. But then an argument can never defend itself. Thus, no singleton set of arguments is admissible.

Even though an abstract view of the notion of attacks between arguments may intuitively impose it as an asymmetric relation, such an intuition is wrong as Constraint 1 yields a quite natural symmetric attack relation.

10. Argumentation with structured news reports

Argumentation has a wide range of application domains, one of which is news reports. Intelligent agents constantly need not only to absorb new information but also to consider the ramifications of it and this implicitly calls for argumentation for identifying pros and cons of various ramifications holding. In the following, we consider the application of argumentation to reasoning with news reports in the form of structured text.

Structured text is an idea implicit in a number of approaches to handling information such as news reports. An item of structured text is a set of semantic labels together with a word, phrase, sentence, null value, or a nested item of structured text, associated with each semantic label. As a simple example, a report on a corporate acquisition could use semantic labels such as "buyer", "seller", "acquisition", "value", and "date".

We assume that news reports are represented as structured text (e.g., in XML or in semi-structured data [1,4] or in a template output from an information extraction system [6,11]) where the text entries are individual words, numbers or very simple phrases. Trials of information extraction systems have been undertaken using various news corpora including news reports and status reports on countries. In Fig. 1, we give a simple example, of the kind of structured text that might be generated by an information extraction system. Since information extraction can incorporate sophisticated lexical information, the output can be given using preferred terms. For example, for `OilPrice`, there may be a number of ways of stating that the oil price is increasing, but it may be preferable to reduce all these alternatives to the entry `increasing`.

An item of structured text such as given in Fig. 1, can be systematically translated into a set of literals. For details on the process and viability of this see [15,16]. For example, we may represent it as follows:

n_0 `Country:x`

n_1 `Date:30 May 2000`

n_2 `Government:unstable`

n_3 `Democracy:strong`

n_4 `PublicSpending:excessive`

n_5 $\texttt{SignificantExport}{:}\texttt{oil}$

n_6 $\texttt{OilPrice}{:}\texttt{increasing}$

n_7 $\texttt{LastElection}{:}\texttt{recent}$

n_8 $\texttt{Currency}{:}\texttt{strong}$

In the domain knowledge we may have numerous formulae that capture possible ramifications of news reports. For illustrations of how some of this domain knowledge might be acquired see [9]. For example, we may have the following formulae.

d_1 $\texttt{Government}{:}\texttt{unstable} \rightarrow \texttt{credit-risk}$

d_2 $\texttt{Democracy}{:}\texttt{strong} \wedge \texttt{LastElection}{:}\texttt{recent} \rightarrow$
$\quad\neg\texttt{Government}{:}\texttt{unstable}$

d_3 $\texttt{PublicSpending}{:}\texttt{excessive} \rightarrow \texttt{credit-risk}$

d_4 $\texttt{Currency}{:}\texttt{strong} \rightarrow \neg\texttt{credit-risk}$

d_5 $\texttt{SignificantExport}{:}\texttt{oil} \wedge \texttt{OilPrice}{:}\texttt{increasing} \rightarrow$
$\quad\neg\texttt{PublicSpending}{:}\texttt{excessive}$

Given a structured news report represented by a set of literals, and a set of formulae representing domain knowledge, we can construct argument structures for ramifications of interest. The argument trees, and the resulting aggregation, provide a means for laying out the relevant facts in the news reports, and associations with the domain knowledge.

```
⟨CountryReport⟩
      ⟨Country⟩ x ⟨/Country⟩
      ⟨Date⟩ 30 May 2000 ⟨/Date⟩
      ⟨Government⟩ unstable ⟨/Government⟩
      ⟨Democracy⟩ strong ⟨/Democracy⟩
      ⟨PublicSpending⟩ excessive ⟨/PublicSpending⟩
      ⟨SignificantExport⟩ oil ⟨/SignificantExport⟩
      ⟨OilPrice⟩ increasing ⟨/OilPrice⟩
      ⟨LastElection⟩ recent ⟨/LastElection⟩
      ⟨Currency⟩ strong ⟨/Currency⟩
⟨/CountryReport⟩
```

Fig. 1. An example of a simple news report in the form of structured text using XML notation.

Now assume the database to be $\Delta = \{n_0, \ldots, n_8, d_1, \ldots, d_5\}$. From this we obtain the following argument trees for and against the inference `credit-risk`.

$$\begin{array}{ccc}
 & \langle\{n_2, d_1\}, \texttt{credit-risk}\rangle & \\
\nearrow & & \nwarrow \\
\langle\{n_3, n_7, d_2\}, \diamond\rangle & & \langle\{n_8, d_4\}, \diamond\rangle \\
 & & \uparrow \\
 & & \langle\{n_4, d_3\}, \diamond\rangle \\
 & & \uparrow \\
 & & \langle\{n_5, n_6, d_5\}, \diamond\rangle
\end{array}$$

$$\begin{array}{ccc}
 & \langle\{n_4, d_3\}, \texttt{credit-risk}\rangle & \\
\nearrow & & \nwarrow \\
\langle\{n_5, n_6, d_5\}, \diamond\rangle & & \langle\{n_8, d_4\}, \diamond\rangle \\
 & & \uparrow \\
 & & \langle\{n_2, d_1\}, \diamond\rangle \\
 & & \uparrow \\
 & & \langle\{n_3, n_7, d_2\}, \diamond\rangle
\end{array}$$

$$\begin{array}{ccc}
 & \langle\{n_8, d_4\}, \neg\texttt{credit-risk}\rangle & \\
\nearrow & & \nwarrow \\
\langle\{n_2, d_1\}, \diamond\rangle & & \langle\{n_4, d_3\}, \diamond\rangle \\
\uparrow & & \uparrow \\
\langle\{n_3, n_7, d_2\}, \diamond\rangle & & \langle\{n_5, n_6, d_5\}, \diamond\rangle
\end{array}$$

Given the simple nature of the database (i.e., the literals obtained from the news report together with the domain knowledge), we see that each tree is a different arrangement of the same set of arguments. However, we stress that each tree is a stand-alone representation of the set of arguments from a particular perspective and so it is necessary to have all these trees.

11. Discussion

In this paper, we have proposed a new framework for modelling argumentation. The key features of this framework are the clarification of the nature of arguments and counter-arguments, the identification of canonical undercuts which we argue are the only defeaters that we need to take into account, and the representation of argument trees and argument structures which provide a way of exhaustively collating arguments and counter-arguments.

This framework can be viewed as a generalization of a wide range of existing approaches to argument aggregation. Moreover, non-binary argument aggregation offers a more realistic approach to weighing up the relative merits of arguments for and against a possible conclusion. Furthermore, there is a range of possible applications of this framework in reasoning with potentially inconsistent information. These include reasoning with inconsistent specifications [12,13], and inconsistent structured text [15,16].

In order to use the framework more generally, we may wish to differentiate the information in Δ from some background knowledge Σ where we assume that Σ is uncontroversial knowledge that can be taken for granted and Δ is controversial knowledge that needs to be regarded as questionable. We can then generalize our definition of the consequence relation $\vdash$ to that of a consequence relation $\vdash_\Sigma$ where inferences can be derived with the benefit of the formulae in Σ.

Finally, we consider capturing a class of arguments that fail to be deductive. For this, the basic principle for our approach still applies: An argument comes with a claim, which relies on reasons by virtue of some given relationship between the reasons and the claim. So, arguments can still be represented by pairs but the relationship is no longer entailment in classical logic, it is a binary relation of some kind capturing "tentative proofs" or "proofs using non-standard modes of inference" instead of logical proofs. This relationship can be taken to be almost whatever pleases you provided that you have a notion of consistency. Observe that this does not mean that you need any second element of a pair to stand for "absurdity": You simply have to specify a subset of the pairs to form the cases of inconsistency. Similarly, our approach is not necessarily restricted to a logical language and another mode of representation can be chosen.

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