Z calculus

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Contents

Introduction

The original source of information is [1]. This text does not claim to have any official authority or originality.

1 Propositional Logic

1.1 Conjunction

Applicable rules:

$$\frac{p \quad q}{p \wedge q}\,[\wedge\!-\!\text{intro}] \qquad\qquad \frac{p \wedge q}{p}\,[\wedge\!-\!\text{elim1}] \qquad\qquad \frac{p \wedge q}{q}\,[\wedge\!-\!\text{elim2}]$$

These three rules say that:

1. From $T \models p$ and $T \models q$ follows also that $T \models p \wedge q$.
2. From $T \models p \wedge q$ follows also that $T \models p$.
3. From $T \models p \wedge q$ follows also that $T \models q$.

Example 1 The fact that conjunction commutes

$$\frac{p \wedge q}{q \wedge p}\,[\wedge\!-\!\text{comm}]$$

can be proved in Z. The set of the assumptions has one item and that item is proposition $p \wedge q$. We want to prove proposition $q \wedge p$. Thus, we start with something like this:

$$\begin{array}{c} p \wedge q \\ \vdots \\ q \wedge p \end{array}$$

We see that we know what assumptions can be used and to what conclusion we want to go. We have to have to make one step toward the complete prooftree[1]. It is *natural* and *obvious* that we have to introduce the bottom conjunction.

$$\frac{\begin{array}{c} p \wedge q \\ \vdots \\ q \end{array} \qquad \begin{array}{c} p \wedge q \\ \vdots \\ p \end{array}}{q \wedge p}\,[\wedge\!-\!\text{intro}]$$

Now we have to prove two propositions: q and p. To do that, we can use any proposition from the set of assumptions (in this case that set has only one element: $p \wedge q$. This fact is also visible in the incomplete prooftree above.

It is also obvious how we should proceed. We are able to prove q directly from $p \wedge q$.

$$\frac{\dfrac{p \wedge q}{q}\,[\wedge\!-\!\text{elim2}] \qquad \begin{array}{c} p \wedge q \\ \vdots \\ p \end{array}}{q \wedge p}\,[\wedge\!-\!\text{intro}]$$

[1] To be able to find the sequence of such steps is part of the *ars inviendi* [2]. This skill/gift has nothing to do with any of the formal logical systems. It is more or less inherent to us all.

The same can be done with p.

$$\dfrac{\dfrac{p \wedge q}{q}[\wedge\!-\mathrm{elim2}] \qquad \dfrac{p \wedge q}{p}[\wedge\!-\mathrm{elim1}]}{q \wedge p}[\wedge\!-\mathrm{intro}]$$

Another prooftree completed purely from the given three rules related to conjunction can be found in Example 10.

Example 2 It is possible to prove that conjunction is an idempotent operation

$$\genfrac{}{}{}{}{p}{p \wedge p}\!\!\!\!\!\!\!\!\genfrac{}{}{}{}{}{}\ [\wedge\!-\mathrm{idempot}]$$

The proof-tree is of $\downarrow$ direction is:

$$\dfrac{p \quad p}{p \wedge p}[\wedge\!-\mathrm{intro}]$$

The proof-tree of $\uparrow$ direction is:

$$\dfrac{p \wedge p}{p}[\wedge\!-\mathrm{elim1}]$$

1.2 Disjunction

Applicable rules:

$$\dfrac{p}{p \vee q}[\vee\!-\mathrm{intro1}] \qquad \dfrac{q}{p \vee q}[\vee\!-\mathrm{intro2}] \qquad \dfrac{p \vee q \qquad \begin{matrix}\lceil p \rceil^{[i]} \\ r\end{matrix} \qquad \begin{matrix}\lceil q \rceil^{[i]} \\ r\end{matrix}}{r}[\vee\!-\mathrm{elim}^{[\mathrm{i}]}]$$

These three rules say that:

1. From $T \models p$ follows also that $T \models p \vee q$.
2. From $T \models q$ follows also that $T \models p \vee q$.
3. From $T \models p \vee q$ and $T, p \models r$ and $T, q \models r$ follows also that $T \models r$.

Example 3 The fact that disjunction commutes

$$\frac{p \vee q}{q \vee p}\,[\vee\!-\!\mathrm{comm}]$$

can be proved as follows

$$\frac{p \vee q \qquad \dfrac{\lceil p \rceil^{[1]}}{q \vee p}\,[\vee\!-\!\mathrm{intro2}] \qquad \dfrac{\lceil q \rceil^{[1]}}{q \vee p}\,[\vee\!-\!\mathrm{intro1}]}{q \vee p}\,[\vee\!-\!\mathrm{elim}^{[1]}]$$

Example 11 contains a prooftree which does not employ different rules but those introduced at the beginning of this section.

Example 4 It is possible to prove that disjunction is an idempotent operation:

$$\begin{array}{c} p \\ \hline\hline p \vee p \end{array}\,[\vee\!-\!\mathrm{idempot}]$$

The proof-tree of the $\downarrow$ direction:

$$\frac{p}{p \vee p}\,[\vee\!-\!\mathrm{intro1}]$$

The proof-tree of the $\uparrow$ direction:

$$\frac{p \vee p \quad \lceil p \rceil^{[1]} \quad \lceil p \rceil^{[2]}}{p}\,[\vee\!-\!\mathrm{elim}^{[1]}]$$

1.3 Implication

Applicable rules:

$$\frac{p \quad p \Rightarrow q}{q}\ [\Rightarrow\text{-elim}] \qquad\qquad \frac{\begin{array}{c}\lceil p \rceil^{[i]} \\ q\end{array}}{p \Rightarrow q}\ [\Rightarrow\text{-intro}^{[i]}]$$

These two rules say that:

1. From $T \models p$ and $T \models p \Rightarrow q$ follows also that $T \models q$.
2. From $T, p \models q$ follows also that $T \models p \Rightarrow q$.

Example 5 We can replace a conjunction of antecedents in an implication by separate antecedents: $(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$. To see how this can be established, consider the incomplete proof tree:

$$\begin{array}{c} \vdots \\ (p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r)) \end{array}$$

$$\frac{\begin{array}{c}\lceil p \wedge q \Rightarrow r \rceil^{[1]} \\ \vdots \\ p \Rightarrow (q \Rightarrow r)\end{array}}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\ [\Rightarrow\text{-intro}^{[1]}]$$

$$\frac{\dfrac{\begin{array}{c}\lceil p \wedge q \Rightarrow r \rceil^{[1]} \\ \lceil p \rceil^{[2]} \\ \vdots \\ q \Rightarrow r\end{array}}{p \Rightarrow (q \Rightarrow r)}\ [\Rightarrow\text{-intro}^{[2]}]}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\ [\Rightarrow\text{-intro}^{[1]}]$$

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\lceil p \wedge q \Rightarrow r \rceil^{[1]}\\ \lceil p \rceil^{[2]}\\ \lceil q \rceil^{[3]}\\ \vdots\\ r\end{array}}{q \Rightarrow r}\ [\Rightarrow\text{–intro}^{[3]}]}{p \Rightarrow (q \Rightarrow r)}\ [\Rightarrow\text{–intro}^{[2]}]}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\ [\Rightarrow\text{–intro}^{[1]}]$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\lceil p \rceil^{[2]}\\ \lceil q \rceil^{[3]}\\ \vdots\\ p \wedge q\end{array} \qquad \lceil p \wedge q \Rightarrow r \rceil^{[1]}}{r}\ [\Rightarrow\text{–elim}]}{q \Rightarrow r}\ [\Rightarrow\text{–intro}^{[3]}]}{p \Rightarrow (q \Rightarrow r)}\ [\Rightarrow\text{–intro}^{[2]}]}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\ [\Rightarrow\text{–intro}^{[1]}]$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\lceil p \rceil^{[2]} \quad \lceil q \rceil^{[3]}}{p \wedge q}\ [\wedge-\text{intro}] \qquad \lceil p \wedge q \Rightarrow r \rceil^{[1]}}{r}\ [\Rightarrow\text{–elim}]}{q \Rightarrow r}\ [\Rightarrow\text{–intro}^{[3]}]}{p \Rightarrow (q \Rightarrow r)}\ [\Rightarrow\text{–intro}^{[2]}]}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\ [\Rightarrow\text{–intro}^{[1]}]$$

1.4 Equivalence

Applicable rules are:

$$\frac{p \Rightarrow q \quad q \Rightarrow p}{p \Leftrightarrow q}\ [\Leftrightarrow\text{–intro}] \qquad \frac{p \Leftrightarrow q}{p \Rightarrow q}\ [\Leftrightarrow\text{–elim1}] \qquad \frac{p \Leftrightarrow q}{q \Rightarrow p}\ [\Leftrightarrow\text{–elim2}]$$

These three rules say that:

1. From $T \models p \Rightarrow q$ and $T \models q \Rightarrow p$ follows also that $T \models p \Leftrightarrow q$.
2. From $T \models p \Leftrightarrow q$ follows also that $T \models p \Rightarrow q$.
3. From $T \models p \Leftrightarrow q$ follows also that $T \models q \Rightarrow p$.

Example 6 If p is stronger than q, then $p \wedge q$ have the same strength. I.e.

$$\frac{p \Rightarrow q}{p \wedge q \Leftrightarrow p}\,[\text{subsume}]$$

It can be proved as follows:

$$\begin{array}{c} p \Rightarrow q \\ \vdots \\ p \wedge q \Leftrightarrow p \end{array}$$

$$\frac{\begin{array}{c} p \Rightarrow q \\ \vdots \\ p \wedge q \Rightarrow q \end{array} \qquad \begin{array}{c} p \Rightarrow q \\ \vdots \\ p \Rightarrow p \wedge q \end{array}}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow\text{-intro}]$$

$$\frac{\dfrac{\begin{array}{c} p \Rightarrow q \\ \lceil p \wedge q \rceil^{[1]} \\ \vdots \\ q \end{array}}{p \wedge q \Rightarrow q}\,[\Rightarrow\text{-intro}^{[1]}] \qquad \dfrac{\begin{array}{c} p \Rightarrow q \\ \lceil p \rceil^{[2]} \\ \vdots \\ p \wedge q \end{array}}{p \Rightarrow p \wedge q}\,[\Rightarrow\text{-intro}^{[2]}]}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow\text{-intro}]$$

Note that the sets of the original assumptions was enriched differently above the sub-goals q and $p \wedge q$.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}p\Rightarrow q\\ \lceil p\wedge q\rceil^{[1]}\\ \vdots\\ p\end{matrix}\qquad p\Rightarrow q}{q}\,[\Rightarrow\text{–elim}]}{p\wedge q\Rightarrow q}\,[\Rightarrow\text{–intro}^{[1]}]\qquad\qquad \dfrac{\begin{matrix}p\Rightarrow q\\ \lceil p\rceil^{[2]}\\ \vdots\\ p\wedge q\end{matrix}}{p\Rightarrow p\wedge q}\,[\Rightarrow\text{–intro}^{[2]}]}{p\wedge q\Leftrightarrow p}\,[\Leftrightarrow\text{–intro}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\lceil p\wedge q\rceil^{[1]}}{p}\,[\wedge-\text{elim1}]\qquad p\Rightarrow q}{q}\,[\Rightarrow\text{–elim}]}{p\wedge q\Rightarrow q}\,[\Rightarrow\text{–intro}^{[1]}]\qquad\qquad \dfrac{\begin{matrix}p\Rightarrow q\\ \lceil p\rceil^{[2]}\\ \vdots\\ p\wedge q\end{matrix}}{p\Rightarrow p\wedge q}\,[\Rightarrow\text{–intro}^{[2]}]}{p\wedge q\Leftrightarrow p}\,[\Leftrightarrow\text{–intro}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\lceil p\wedge q\rceil^{[1]}}{p}\,[\wedge-\text{elim1}]\qquad p\Rightarrow q}{q}\,[\Rightarrow\text{–elim}]}{p\wedge q\Rightarrow q}\,[\Rightarrow\text{–intro}^{[1]}]\qquad\qquad \dfrac{\dfrac{\begin{matrix}p\Rightarrow q\\ \lceil p\rceil^{[2]}\\ \vdots\\ p\end{matrix}\qquad\begin{matrix}p\Rightarrow q\\ \lceil p\rceil^{[2]}\\ \vdots\\ q\end{matrix}}{p\wedge q}\,[\wedge-\text{intro}]}{p\Rightarrow p\wedge q}\,[\Rightarrow\text{–intro}^{[2]}]}{p\wedge q\Leftrightarrow p}\,[\Leftrightarrow\text{–intro}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\lceil p\wedge q\rceil^{[1]}}{p}\,[\wedge-\text{elim1}]\qquad p\Rightarrow q}{q}\,[\Rightarrow\text{–elim}]}{p\wedge q\Rightarrow q}\,[\Rightarrow\text{–intro}^{[1]}]\qquad\qquad \dfrac{\dfrac{\lceil p\rceil^{[2]}\qquad\begin{matrix}p\Rightarrow q\\ \lceil p\rceil^{[2]}\\ \vdots\\ q\end{matrix}}{p\wedge q}\,[\wedge-\text{intro}]}{p\Rightarrow p\wedge q}\,[\Rightarrow\text{–intro}^{[2]}]}{p\wedge q\Leftrightarrow p}\,[\Leftrightarrow\text{–intro}]$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\lceil p \wedge q \rceil^{[1]}}{p}\,[\wedge-\mathrm{elim1}]
\qquad p \Rightarrow q
}{q}\,[\Rightarrow-\mathrm{elim}]
}{p \wedge q \Rightarrow q}\,[\Rightarrow-\mathrm{intro}^{[1]}]
\qquad
\dfrac{
\dfrac{
\lceil p \rceil^{[2]}
\qquad
\dfrac{\lceil p \rceil^{[2]} \qquad p \Rightarrow q}{q}\,[\Rightarrow-\mathrm{elim}]
}{p \wedge q}\,[\wedge-\mathrm{intro}]
}{p \Rightarrow p \wedge q}\,[\Rightarrow-\mathrm{intro}^{[2]}]
}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow-\mathrm{intro}]
$$

1.5 Negation

Applicable rules are:

$$\frac{\begin{array}{c}\lceil p \rceil^{[i]} \\ false\end{array}}{\neg p}\,[\neg\text{−intro}^{[i]}] \qquad \frac{p \quad \neg p}{false}\,[\neg\text{−elim}] \qquad \frac{\begin{array}{c}\lceil \neg p \rceil^{[j]} \\ false\end{array}}{p}\,[false\text{−elim}^{[j]}]$$

These rules say that:

1. From $T, p \models \bot$ follows also that $T \models \neg p$.
2. From $T \models p$ and $T \models \neg p$ follows also that $T \models \bot$.
3. From $T, \neg p \models \bot$ follows also that $T \models p$.

Example 7 The following inference rule:

$$\frac{\neg\neg p}{\overline{p}}\,[\neg\neg\text{−elim}]$$

can be proved as follows: $\downarrow$ direction

$$\dfrac{\dfrac{\neg\neg p \quad \lceil \neg p \rceil^{[1]}}{false}\,[\neg\text{−elim}]}{p}\,[false\text{−elim}^{[1]}]$$

and: $\uparrow$ direction

$$\dfrac{\dfrac{p \quad \lceil \neg p \rceil^{[1]}}{false}\,[\neg\text{−elim}]}{\neg\neg p}\,[\neg\text{−intro}^{[1]}]$$

Example 8 One of the de Morgan Laws states that the negation of a disjunction is the conjunction of negations:

$$\frac{\neg(p \vee q)}{\overline{\neg p \wedge \neg q}}\,[\text{deMorgan1}]$$

It can be proved as follows: $\downarrow$ direction

$$\dfrac{\dfrac{\dfrac{\dfrac{\lceil p \rceil^{[1]}}{p \vee q}\,[\vee\!-\!\text{intro1}] \qquad \neg(p \vee q)}{false}\,[\neg\!-\!\text{elim}]}{\neg p}\,[\neg\!-\!\text{intro}^{[1]}] \qquad \dfrac{\dfrac{\dfrac{\lceil q \rceil^{[2]}}{p \vee q}\,[\vee\!-\!\text{intro2}] \qquad \neg(p \vee q)}{false}\,[\neg\!-\!\text{elim}]}{\neg q}\,[\neg\!-\!\text{intro}^{[2]}]}{\neg p \wedge \neg q}\,[\wedge\!-\!\text{intro}]$$

and $\uparrow$ direction:

$$\dfrac{\dfrac{\dfrac{\neg p \wedge \neg q}{\neg p}\,[\wedge\!-\!\text{elim1}] \qquad \dfrac{\dfrac{\lceil \neg\neg(p \wedge q) \rceil^{[1]}}{p \wedge q}\,[\neg\neg\!-\!\text{elim}]}{p}\,[\wedge\!-\!\text{elim1}]}{false}\,[\neg\!-\!\text{elim}]}{\neg(p \wedge q)}\,[false\!-\!\text{elim}^{[1]}]$$

The $\uparrow$ direction could be alternatively proved also as:

$$\dfrac{\dfrac{\dfrac{\neg p \wedge \neg q}{\neg q}\,[\wedge\!-\!\text{elim2}] \qquad \dfrac{\dfrac{\lceil \neg\neg(p \wedge q) \rceil^{[1]}}{p \wedge q}\,[\neg\neg\!-\!\text{elim}]}{q}\,[\wedge\!-\!\text{elim2}]}{false}\,[\neg\!-\!\text{elim}]}{\neg(p \wedge q)}\,[false\!-\!\text{elim}^{[1]}]$$

Example 9 One of the de Morgan Laws states that the negation of a conjunction is the disjunction of negations:

$$\dfrac{\neg(p \wedge q)}{\neg p \vee \neg q}\,[\text{deMorgan2}]$$

It can be proved as follows: $\downarrow$ direction

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil \neg(\neg p \lor \neg q)\rceil^{[1]}}{\neg\neg p \land \neg\neg q}\,[\text{deMorgan1}]}{\neg\neg p}\,[\land-\text{elim1}]}{p}\,[\neg\neg-\text{elim}] \qquad \dfrac{\dfrac{\dfrac{\lceil \neg(\neg p \lor \neg q)\rceil^{[1]}}{\neg\neg p \land \neg\neg q}\,[\text{deMorgan1}]}{\neg\neg q}\,[\land-\text{elim2}]}{q}\,[\neg\neg-\text{elim}]}{p \land q}\,[\land-\text{intro}] \qquad \neg(p \land q)}{false}\,[\neg-\text{elim}]}{\neg p \lor \neg q}\,[false-\text{elim}^{[1]}]
$$

and ↑ direction

$$
\dfrac{\dfrac{\neg p \lor \neg q \qquad \dfrac{\lceil \neg p\rceil^{[2]} \qquad \dfrac{\lceil p \land q\rceil^{[1]}}{p}\,[\land-\text{elim1}]}{false}\,[\neg-\text{elim}] \qquad \dfrac{\lceil \neg q\rceil^{[2]} \qquad \dfrac{\lceil p \land q\rceil^{[1]}}{q}\,[\land-\text{elim2}]}{false}\,[\neg-\text{elim}]}{false}\,[\lor-\text{elim}^{[2]}]}{\neg(p \land q)}\,[\neg-\text{intro}^{[1]}]
$$

Example 10 The fact that conjunction is associative:

$$
\dfrac{(p \land q) \land r}{p \land (q \land r)}\,[\land-\text{assoc}]
$$

can be showed as follows: ↓ direction

$$
\dfrac{\dfrac{\dfrac{(p \land q) \land r}{p \land q}\,[\land-\text{elim1}]}{p}\,[\land-\text{elim1}] \qquad \dfrac{\dfrac{\dfrac{(p \land q) \land r}{p \land q}\,[\land-\text{elim1}]}{q}\,[\land-\text{elim2}] \qquad \dfrac{(p \land q) \land r}{r}\,[\land-\text{elim2}]}{q \land r}\,[\land-\text{intro}]}{p \land (q \land r)}\,[\land-\text{intro}]
$$

and ↑ direction:

$$
\dfrac{\dfrac{\dfrac{p \wedge (q \wedge r)}{p}\,[\wedge\!-\!\text{elim1}] \qquad \dfrac{\dfrac{p \wedge (q \wedge r)}{q \wedge r}\,[\wedge\!-\!\text{elim2}]}{q}\,[\wedge\!-\!\text{elim1}]}{1p \wedge q}\,[\wedge\!-\!\text{intro}] \qquad \dfrac{\dfrac{p \wedge (q \wedge r)}{q \wedge r}\,[\wedge\!-\!\text{elim2}]}{r}\,[\wedge\!-\!\text{elim2}]}{(p \wedge q) \wedge r}\,[\wedge\!-\!\text{intro}]
$$

Example 11 The fact that disjunction is associative:

$$
\dfrac{(p \vee q) \vee r}{\overline{p \vee (q \vee r)}}\,[\vee\!-\!\text{assoc}]
$$

can be showed as follows: $\downarrow$ direction

$$
\dfrac{
\dfrac{
(p \lor q) \lor r
\qquad
\dfrac{
\lceil p \lor q \rceil^{[2]}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\lceil \neg(p \lor (q \lor r)) \rceil^{[1]}}{p \land \neg(q \lor r)}\;[\text{deMorgan}\neg]
}{\neg p}\;[\land{-}\text{elim1}]
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\lceil \neg(p \lor (q \lor r)) \rceil^{[1]}}{\neg p \land \neg(q \lor r)}\;[\text{deMorgan1}]
}{\neg(q \lor r)}\;[\land{-}\text{elim2}]
}{\neg q \land \neg r}\;[\text{deMorgan1}]
}{\neg q}\;[\land{-}\text{elim1}]
}{\neg p \land \neg q}\;[\land{-}\text{intro}]
}{\neg(p \lor q)}\;[\text{deMorgan1}]
}{}
}{false}\;[\neg{-}\text{elim}]
\qquad
\dfrac{
\lceil r \rceil^{[2]}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\lceil \neg(p \lor (q \lor r)) \rceil^{[1]}}{\neg p \land \neg(q \lor r)}\;[\text{deMorgan1}]
}{\neg(q \lor r)}\;[\land{-}\text{elim2}]
}{\neg q \land \neg r}\;[\text{deMorgan1}]
}{\neg r}\;[\land{-}\text{elim2}]
}{false}\;[\neg{-}\text{elim}]
}{false}\;[\lor{-}\text{elim}^{[2]}]
}{p \lor (q \lor r)}\;[false{-}\text{elim}^{[1]}]
$$

The $\uparrow$ direction can be proved as follows:

$$
\dfrac{
\dfrac{
p \lor (q \lor r)
\qquad
\dfrac{
\lceil p \rceil^{[2]}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\lceil \neg((p \lor q) \lor r) \rceil^{[1]}}{\neg(p \lor q) \land \neg r}\;[\text{deMorgan1}]
}{\neg(p \lor q)}\;[\land{-}\text{elim1}]
}{\neg p \land \neg q}\;[\text{deMorgan1}]
}{\neg p}\;[\land{-}\text{elim1}]
}{false}\;[\neg{-}\text{elim}]
\qquad
\dfrac{
\lceil q \lor r \rceil^{[2]}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\lceil \neg((p \lor q) \lor r) \rceil^{[1]}}{\neg(p \land q) \land \neg r}\;[\text{deMorgan1}]
}{\neg(p \land q)}\;[\land{-}\text{elim1}]
}{\neg p \land \neg q}\;[\text{deMorgan1}]
}{\neg q}\;[\land{-}\text{elim2}]
\qquad
\dfrac{
\dfrac{\lceil \neg((p \lor q) \lor r) \rceil^{[1]}}{\neg(p \lor q) \land \neg r}\;[\text{deMorgan1}]
}{\neg r}\;[\land{-}\text{elim2}]
}{\neg q \land \neg r}\;[\land{-}\text{intro}]
}{\neg(q \lor r)}\;[\text{deMorgan1}]
}{false}\;[\neg{-}\text{elim}]
}{false}\;[\lor{-}\text{elim}^{[2]}]
}{(p \lor q) \lor r}\;[false{-}\text{elim}^{[1]}]
$$

Example 12 Tertia non datur:

$$\frac{}{p \vee \neg p}$$

can be proved simply as:

$$\cfrac{\cfrac{\cfrac{\cfrac{\lceil \neg(p \vee \neg q) \rceil^{[1]}}{\neg p \wedge \neg\neg p}\,[\text{deMorgan1}]}{\neg p}\,[\wedge-\text{elim1}] \qquad \cfrac{\cfrac{\lceil \neg(p \vee \neg q) \rceil^{[1]}}{\neg p \wedge \neg\neg p}\,[\text{deMorgan1}]}{\neg\neg p}\,[\wedge-\text{elim2}]}{false}\,[\neg-\text{elim}]}{p \vee \neg p}\,[false-\text{elim}^{[1]}]$$

Here we show proofs of some of the propositions in [**?**] on page 19, exercise 1.1

Example 13 Exercise 19/7 in [**?**]. This:

$$\frac{}{a \wedge a \Leftrightarrow a}$$

can be proved as:

$$\cfrac{\cfrac{\cfrac{\lceil a \wedge a \rceil^{[1]}}{a}\,[\wedge-\text{elim1}]}{a \wedge a \Rightarrow a}\,[\Rightarrow-\text{intro}^{[1]}] \qquad \cfrac{\cfrac{\lceil a \rceil^{[2]} \quad \lceil a \rceil^{[2]}}{a \wedge a}\,[\wedge-\text{intro}]}{a \Rightarrow a \wedge a}\,[\Rightarrow-\text{intro}^{[2]}]}{a \wedge a \Leftrightarrow a}\,[\Leftrightarrow-\text{intro}]$$

Example 14 Exercise 19/8 in [**?**]. This:

$$\frac{}{a \vee a \Leftrightarrow a}$$

can be proved as:

$$
\dfrac{
  \dfrac{
    \dfrac{\lceil a \vee a \rceil^{[1]} \quad \lceil a \rceil^{[2]} \quad \lceil a \rceil^{[2]}}{a}\;[\vee\text{−elim}^{[2]}]
  }{a \vee a \Rightarrow a}\;[\Rightarrow\text{−intro}^{[1]}]
  \qquad
  \dfrac{
    \dfrac{\lceil a \rceil^{[3]}}{a \vee a}\;[\vee\text{−intro1}]
  }{a \Rightarrow a \vee a}\;[\Rightarrow\text{−intro}^{[3]}]
}{a \vee a \Leftrightarrow a}\;[\Leftrightarrow\text{−intro}]
$$

Example 15 Exercise 19/18 in [?]. This:

$$
\dfrac{}{a \vee (a \wedge b) \Leftrightarrow a}
$$

can be proved as:

$$
\dfrac{
  \dfrac{
    \dfrac{\lceil a \vee (a \wedge b) \rceil^{[1]} \quad \lceil a \rceil^{[2]} \quad \dfrac{\lceil a \wedge b \rceil^{[2]}}{a}\;[\wedge\text{−elim1}]}{a}\;[\vee\text{−elim}^{[2]}]
  }{a \vee (a \wedge b) \Rightarrow a}\;[\Rightarrow\text{−intro}^{[1]}]
  \qquad
  \dfrac{
    \dfrac{\lceil a \rceil^{[3]}}{a \vee (a \wedge b)}\;[\vee\text{−intro1}]
  }{a \Rightarrow a \vee (a \wedge b)}\;[\Rightarrow\text{−intro}^{[3]}]
}{a \vee (a \wedge b) \Leftrightarrow a}\;[\Leftrightarrow\text{−intro}]
$$

Example 16 Exercise 20/19 in [?]. This:

$$
\dfrac{}{a \wedge (a \vee b) \Leftrightarrow a}
$$

can be proved as:

$$
\dfrac{
  \dfrac{
    \dfrac{\lceil a \wedge (a \vee b) \rceil^{[1]}}{a}\;[\wedge\text{−elim1}]
  }{a \wedge (a \vee b) \Rightarrow a}\;[\Rightarrow\text{−intro}^{[1]}]
  \qquad
  a \Rightarrow a \wedge (a \vee b)
}{a \wedge (a \vee b) \Leftrightarrow a}\;[\Leftrightarrow\text{−intro}]
$$

Example 17 Exercise 20/20 in [?]. This:

$$\overline{(a \Rightarrow b) \Leftrightarrow (\neg a \lor b)}$$

can be proved as:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil \neg(\neg a \lor \neg b) \rceil^{[2]}}{\neg\neg a \land \neg b}\;[\text{deMorgan1}]}{\neg\neg a}\;[\land-\text{elim1}]}{a}\;[\neg\neg-\text{elim}] \qquad \lceil a \Rightarrow b \rceil^{[1]}}{b}\;[\Rightarrow-\text{elim}] \qquad \dfrac{\begin{matrix}\dfrac{\lceil \neg(\neg a \lor \neg b) \rceil^{[2]}}{\neg\neg a \land \neg b}\;[\text{deMorgan1}] \\ \overline{}\;[\land-\text{elim2}] \\ \vdots \end{matrix}}{\neg b}\;[\land-\text{elim2}]}{false}\;[\neg-\text{elim}]}{\neg a \lor b}\;[false-\text{elim}^{[2]}]}{(a \Rightarrow b) \Rightarrow (\neg a \lor b)}\;[\Rightarrow-\text{intro}^{[1]}] \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{\lceil \neg a \lor b \rceil^{[3]} \quad \dfrac{\dfrac{\lceil a \rceil^{[4]} \quad \lceil \neg a \rceil^{[5]}}{false}\;[\neg-\text{elim}]}{b}\;[false-\text{elim}^{[6]}] \quad \lceil b \rceil^{[5]}}{b}\;[\lor-\text{elim}^{[5]}]}{a \Rightarrow b}\;[\Rightarrow-\text{intro}^{[4]}] \\ \overline{}\;[\Rightarrow-\text{intro}^{[3]}] \\ \vdots \end{matrix}}{(\neg a \lor b) \Rightarrow (a \Rightarrow b)}\;[\Rightarrow-\text{intro}^{[3]}]}{(a \Rightarrow b) \Leftrightarrow (\neg a \lor b)}\;[\Leftrightarrow-\text{intro}]$$

2 Predicate Logic

2.1 Substitution

By substitution we mean replacing all free occurrences of a given substitutee (some variable) by a given substituent (some term). There are certain rules we must obey when doing substitution.

We can give equivalences to explain the effect of substitution into quantified expressions. In the simplest case, the variable being substituted for has the same name as the one being quantified:

$$\begin{array}{lcl}(\forall x : a \bullet p)[t \,/\, x] & \Leftrightarrow & (\forall x : a \bullet p) \\ (\exists x : a \bullet p)[t \,/\, x] & \Leftrightarrow & (\exists x : a \bullet p)\end{array}$$

Substitution affects only free occurrences of a given variable. All occurrences of the x variable above are bound.

Example 18 Consider the predicate:

$$\exists p : Person \bullet \neg(p\ LooksLike\ o)$$

Substitution of the term m instead of all free occurrences of p results in

$$\exists p : Person \bullet \neg(p\ LooksLike\ o)$$

I.e. the predicate will be unchanged because the variable being replaced is the one which is being quantified.

If the quantifier is binding some variable other than x, then the substitution will have more of an effect. If y is not free (it does not occur as free variable) in t, then then

$$\begin{array}{lcl} (\forall y : a \bullet p)[t \,/\, x] & \Leftrightarrow & (\forall y : a \bullet p[t \,/\, x]) \\ (\exists y : a \bullet p)[t \,/\, x] & \Leftrightarrow & (\exists y : a \bullet p[t \,/\, x]) \end{array}$$

Example 19 Consider the predicate:

$$\exists p : Person \bullet \neg(p\ LooksLike\ o)$$

Substitution of the term m instead of all free occurrences of o results in

$$\exists p : Person \bullet \neg(p\ LooksLike\ m)$$

The predicate changed in a desired way.

If y is free (occurs as free variable) in t, then we choose a fresh variable z, different from x and not appearing in t:

$$\begin{array}{lcl} (\forall y : a \bullet p)[t \,/\, x] & \Leftrightarrow & (\forall z : a \bullet p[z \,/\, y][t \,/\, x]) \\ (\exists y : a \bullet p)[t \,/\, x] & \Leftrightarrow & (\exists y : a \bullet p[z \,/\, y][t \,/\, x]) \end{array}$$

By using z instead of y for the name of the quantified variable, we have avoided any possibility of unintentional variable capture (of the originally free variables occurring in t).

Example 20 Consider the predicate:

$$\exists p : Person \bullet \neg(p\ LooksLike\ o)$$

Substitution of the term p instead of all free occurrences of o cannot be done as:

$$\exists p : Person \bullet \neg(p\ LooksLike\ p)$$

because originally free variable p becomes captured and thus it would change the original meaning of the predicate in an undesired way.

The rule above says that since p (quantified variable) occurs free in p (substituent) we have first to change the quantified variable p to some fresh variable (say z) and then substitute a given substituent instead of a given substitutee. Thus, we get:

$$\exists z : Person \bullet \neg(z\ LooksLike\ p)$$

If the major operator in an expression is not a quantifier, then the effect of substitution is easy to explain:

$$\begin{array}{rcl} (\neg p)[t\,/\,x] & \Leftrightarrow & \neg p[t\,/\,x] \\ (p \wedge q)[t\,/\,x] & \Leftrightarrow & p[t\,/\,x] \wedge q[t\,/\,x] \\ (p \vee q)[t\,/\,x] & \Leftrightarrow & p[t\,/\,x] \vee q[t\,/\,x] \\ (p \Rightarrow q)[t\,/\,x] & \Leftrightarrow & p[t\,/\,x] \Rightarrow q[t\,/\,x] \\ (p \Leftrightarrow q)[t\,/\,x] & \Leftrightarrow & p[t\,/\,x] \Leftrightarrow q[t\,/\,x] \end{array}$$

2.2 Universal Introduction and Elimination

In general, the truth-table technique for giving meaning to connectives and reasoning about them is useless for the quantifiers, since the sets that bound variables may range over are simply too large. However, we may build upon the natural deduction system of the previous chapter by adding rules to introduce and eliminate quantifiers.

If we view universal quantification as a generalized conjunction, then we should be able to generalize the rules for conjunction to get the rules for the universal quantifier. Consider first the introduction rule. In order to prove $p \wedge q$, one needs to prove both p and q. In order to prove $\forall x : a \bullet q$, one must prove that p is true for each value in a. This doesn't sound terribly hopeful, as it might involve an infinite number of premises, and therefore an infinite number of proofs.

A better approach might be to prove that p holds for an arbitrary member of a: if we make no assumptions whatsoever about which member of a we choose in order to prove p, then our proof generalizes to all members.

$$\frac{\begin{array}{c}\lceil x \in a \rceil^{[i]} \\ p\end{array}}{\forall x : a \bullet p}\ [\forall\!-\!\mathrm{intro}^{[\mathrm{i}]}]$$

This rule is applicable only if x is not free (does not occur as free variable) in the assumptions of p. If x appeared free in any of the assumptions used to prove p, that could enable us to prove p under these constraints even though p does not hold for any value taken from a. Only with the above additional requirement can be the above rule correct. In other words:

- From $T, x \in a \models p$ follows also that $T \models \forall x : a \bullet p$.

From conjunction, one may conclude either of the conjuncts; by analogy, from a universally quantified predicate, one may conclude that the predicate holds for any value in the range. Suppose that we have the universally quantified predicate $\forall x : a \bullet p$, and that the expression t denotes a value from a; then p must be true of t.

$$\frac{t \in a \quad \forall x : a \bullet p}{p[t \,/\, x]} \; [\forall{-}\mathrm{elim}]$$

In other words:

- From $T \models x \in a$ and $T \models \forall x : a \bullet p$ follows also that $T \models p[t \,/\, x]$

A special case of the last rule takes t as x:

$$\frac{x \in a \quad \forall x : a \bullet p}{p} \; [\forall{-}\mathrm{elim}]$$

Example 21 The universal quantifier distributes through conjunction. We will prove this in the one direction only.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil x \in a \rceil^{[2]} \quad \lceil \forall x : a \bullet p \wedge q \rceil^{[1]}}{p \wedge q}\,[\forall{-}\mathrm{elim}]}{p}\,[\wedge{-}\mathrm{elim1}]}{\forall x : a \bullet p}\,[\forall{-}\mathrm{intro}^{[2]}] \qquad \dfrac{\dfrac{\dfrac{\lceil x \in a \rceil^{[3]} \quad \lceil \forall x : a \bullet p \wedge q \rceil^{[1]}}{p \wedge q}\,[\forall{-}\mathrm{elim}]}{q}\,[\wedge{-}\mathrm{elim2}]}{\forall x : a \bullet q}\,[\forall{-}\mathrm{intro}^{[3]}]}{(\forall x : a \bullet p) \wedge (\forall x : a \bullet q)}\,[\wedge{-}\mathrm{intro}]}{(\forall x : a \bullet p \wedge q) \Rightarrow ((\forall x : a \bullet p) \wedge (\forall x : a \bullet q))}\,[\Rightarrow{-}\mathrm{intro}^{[1]}]$$

2.3 Existential Introduction and Elimination

To introduce an existential quantifier, we must show that a suitable expression t exists: we must provide an example.

$$\frac{t \in a \quad p[t\,/\,x]}{\exists x : a \bullet p}\,[\exists\!-\!\text{intro}]$$

In other words:

- From $T \models t \in a$ and $T \models p[t\,/\,x]$ follows also that $T \models \exists x : a \bullet p$.

Elimination of the existential quantifier is a more difficult affair. The predicate $\exists x : a \bullet s$ states that there is some object x in a for which s is true. If x appears free in p then simply removing the quantifier leaves us with an unjustified statement about a free variable x. We cannot, in general, conclude p from $\exists x : a \bullet p$. To use the information contained in p, we must complete any reasoning that involves x before eliminating the quantifier.

Suppose that we assume only that $x \in a$ and that p holds of x. If we are then able to derive a predicate r that does not involve x, and we know that there is some x in a for which p is true, then we may safely conclude r.

$$\frac{\exists x : a \bullet p \qquad \begin{matrix}\lceil x \in a \wedge p \rceil^{[i]} \\ r\end{matrix}}{r}\,[\exists\!-\!\text{elim}^{[\text{i}]}]$$

In other words:

- From $T \models \exists x : a \bullet p$ and $T, x \in a, p \models r$ follows also that $T \models r$.

This rule is applicable only if x is not free in the assumptions, and x is not free in r. It is important that nothing is assumed about x during the derivation of r, apart from the explicit assumptions: $x \in a$ and p.

Example 22 The rule:

$$\frac{\neg(\exists x : a \bullet p)}{\overline{\forall x : a \bullet \neg p}}\,[\text{deMorgan3}]$$

can be proved as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\lceil x \in a \rceil^{[1]} \quad \lceil p \rceil^{[2]}}{\exists x : a \bullet p}\,[\exists-\text{intro}] \qquad \neg(\exists x : a \bullet p)}{false}\,[\neg-\text{elim}]}{\neg p}\,[\neg-\text{intro}^{[2]}]}{\forall x : a \bullet \neg p}\,[\forall-\text{intro}^{[1]}]$$

Example 23 The fact that:

$$\dfrac{\neg(\forall x : a \bullet p)}{\exists x : a \bullet \neg p}\,[\text{deMorgan4}]$$

can be proved as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil(\exists x : a \bullet \neg p)\rceil^{[1]}}{\forall x : a \bullet \neg\neg p}\,[\text{deMorgan3}] \qquad \lceil x \in a \rceil^{[2]}}{\neg\neg p}\,[\forall-\text{elim}]}{p}\,[\neg\neg-\text{elim}]}{\forall x : a \bullet p}\,[\forall-\text{intro}^{[2]}] \qquad \neg(\forall x : a \bullet p)}{false}\,[\neg-\text{elim}]}{\exists x : a \bullet \neg p}\,[false-\text{elim}^{[1]}]$$

2.4 Couples as bound variables

Here we introduce (for some occasions convenient) possibility to use couples as variables bound by quantifier or by set comprehension.

Suppose that $p(x, y)$ and $q(x, y)$ are predicates with free variables x and y. Then:

$$\begin{aligned}
\forall(x, y) : A \bullet p(x, y) &\equiv \forall c : A \bullet p(c.1, c.2) \\
\forall(x, y) : A \mid p(x, y) \bullet q(x, y) &\equiv \forall c : A \mid p(c.1, c.2) \bullet q(c.1, c.2) \\
\exists(x, y) : A \bullet p(x, y) &\equiv \exists c : A \bullet p(c.1, c.2) \\
\exists(x, y) : A \mid p(x, y) \bullet q(x, y) &\equiv \exists c : A \mid p(c.1, c.2) \bullet q(c.1, c.2)
\end{aligned}$$

3 Sets

All the necessary rules are given in [1]. We will not repeat them here.

Additional abbreviations:

$$\begin{array}{rcl} R\,\mathrm{proj}\,i & == & \{r : R \bullet r.i\} \\ R^{-1} & == & \{c : (R\,\mathrm{proj}\,2\,) \times (R\,\mathrm{proj}\,1\,) \mid (c.2, c.1) \in R\} \end{array}$$

Example 24 It is possible to prove that union is an idempotent operation:

$$\dfrac{}{A \cup A = A}\,[\cup-\text{idempot}]$$

as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil x \in A \cup A \rceil^{[1]}}{x \in A \lor x \in A}\,[\text{union}]}{x \in A}\,[\lor-\text{idempot}]}{\forall x : A \cup A \bullet x \in A}\,[\forall-\text{intro}^{[1]}] \qquad \dfrac{\dfrac{\dfrac{\lceil x \in A \rceil^{[2]}}{x \in A \lor x \in A}\,[\lor-\text{idempot}]}{x \in A \cup A}\,[\text{union}]}{\forall x : A \bullet x \in A \cup A}\,[\forall-\text{intro}^{[2]}]}{(\forall x : A \cup A \bullet x \in A) \land (\forall x : A \bullet x \in A \cup A)}\,[\land-\text{intro}]}{A \cup A = A}\,[\text{ext}]$$

Example 25 It is possible to prove that intersection is an idempotent operation:

$$\dfrac{}{A \cap A = A}\,[\cap-\text{idempot}]$$

as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil x \in A \cap A \rceil^{[1]}}{x \in A \land x \in A}\,[\text{union}]}{x \in A}\,[\land-\text{idempot}]}{\forall x : A \cap A \bullet x \in A}\,[\forall-\text{intro}^{[1]}] \qquad \dfrac{\dfrac{\dfrac{\lceil x \in A \rceil^{[2]}}{x \in A \land x \in A}\,[\land-\text{idempot}]}{x \in A \cap A}\,[\text{union}]}{\forall x : A \bullet x \in A \cap A}\,[\forall-\text{intro}^{[2]}]}{(\forall x : A \cap A \bullet x \in A) \land (\forall x : A \bullet x \in A \cap A)}\,[\land-\text{intro}]}{A \cup A = A}\,[\text{ext}]$$

Example 26 It is possible to prove that intersection is a commutative operation

$$\dfrac{}{A \cap B = B \cap A}\,[\cap\!-\!\mathrm{comm}]$$

as follows

$$
\begin{prooftree}
\AxiomC{$\lceil x \in B \cap A \rceil^{[1]}$}
\doubleLine
\RightLabel{[inter]}
\UnaryInfC{$x \in B \wedge x \in A$}
\RightLabel{[$\wedge-$comm]}
\UnaryInfC{$x \in A \wedge x \in B$}
\doubleLine
\RightLabel{[inter]}
\UnaryInfC{$x \in B \cap A$}
\RightLabel{[$\forall-$intro$^{[1]}$]}
\UnaryInfC{$\forall x : A \cap B \bullet x \in B \cap A$}
\AxiomC{$\lceil x \in A \cap B \rceil^{[2]}$}
\doubleLine
\RightLabel{[inter]}
\UnaryInfC{$x \in A \wedge x \in B$}
\RightLabel{[$\wedge-$comm]}
\UnaryInfC{$x \in B \wedge x \in A$}
\doubleLine
\RightLabel{[inter]}
\UnaryInfC{$x \in A \cap B$}
\RightLabel{[$\forall-$intro$^{[2]}$]}
\UnaryInfC{$\forall x : B \cap A \bullet x \in A \cap B$}
\RightLabel{[$\wedge-$intro]}
\BinaryInfC{$(\forall x : A \cap B \bullet x \in B \cap A) \wedge (\forall x : B \cap A \bullet x \in A \cap B)$}
\doubleLine
\RightLabel{[ext]}
\UnaryInfC{$A \cap B = B \cap A$}
\end{prooftree}
$$

Example 27 It is possible to prove that union is a commutative operation

$$\dfrac{}{A \cup B = B \cup A}\,[\cup\!-\!\mathrm{comm}]$$

as follows:

$$
\begin{prooftree}
\AxiomC{$\lceil x \in A \cup B \rceil^{[1]}$}
\doubleLine
\RightLabel{[union]}
\UnaryInfC{$x \in A \vee x \in B$}
\RightLabel{[$\vee-$comm]}
\UnaryInfC{$x \in B \vee x \in A$}
\doubleLine
\RightLabel{[union]}
\UnaryInfC{$x \in B \cup A$}
\RightLabel{[$\forall-$intro$^{[1]}$]}
\UnaryInfC{$\forall x : A \cup B \bullet x \in B \cup A$}
\AxiomC{$\lceil x \in B \cup A \rceil^{[2]}$}
\doubleLine
\RightLabel{[union]}
\UnaryInfC{$x \in B \vee x \in A$}
\RightLabel{[$\vee-$comm]}
\UnaryInfC{$x \in A \vee x \in B$}
\doubleLine
\RightLabel{[union]}
\UnaryInfC{$x \in A \cup B$}
\RightLabel{[$\forall-$intro$^{[2]}$]}
\UnaryInfC{$\forall x : B \cup A \bullet x \in A \cup B$}
\RightLabel{[$\wedge-$intro]}
\BinaryInfC{$(\forall x : A \cup B \bullet x \in B \cup A) \wedge (\forall x : B \cup A \bullet x \in A \cup B)$}
\doubleLine
\RightLabel{[ext]}
\UnaryInfC{$A \cup B = B \cup A$}
\end{prooftree}
$$

Example 28 Its possible to prove

$$\frac{x \in A \qquad\qquad A \subseteq B}{x \in B}\,[\text{subset-mem}]$$

as follows:

$$\frac{x \in A \qquad\qquad \dfrac{A \subseteq B}{\forall y : A \bullet y \in B}\,[\text{subset}]}{x \in B}\,[\forall{-}\text{elim}]$$

Example 29 It is possible to prove

$$\frac{p[t/x]}{p[(t.1, t.2)/x]} \quad \downarrow [.\ \text{intro}] \quad \uparrow [.\ \text{elim}]$$

as follows:

$$\frac{\dfrac{\dfrac{\dfrac{}{t.1 = t.1}\,[\text{eq-ref}] \qquad \dfrac{}{t.2 = t.2}\,[\text{eq-ref}]}{t = (t.1, t.2)}\,[\text{cart-proj}]}{(t.1, t.2) = t}\,[\text{eq-sym}] \qquad\qquad p[t/x]}{p[(t.1, t.2)/x]}\,[\text{eq-sub}]$$

Example 30 Its possible to prove

$$\frac{}{\mathrm{id}\, A \subseteq A \times A}\,[\text{id-sub-cart}]$$

as follows:

$$
\begin{prooftree}
\AxiomC{$\lceil x \in \operatorname{id} A \rceil^{[1]}$}
\RightLabel{[abbrev]}
\UnaryInfC{$x \in y : A \bullet (y, y)$}
\RightLabel{[compre]}
\UnaryInfC{$\exists y : A \bullet (y, y) = x$}
\AxiomC{$\lceil y \in A \rceil^{[2]}$}
\AxiomC{$\lceil y \in A \rceil^{[2]}$}
\RightLabel{[cart-mem]}
\BinaryInfC{$(y, y) \in A \times A$}
\AxiomC{$\lceil (y, y) = x \rceil^{[2]}$}
\RightLabel{[eq-sub]}
\BinaryInfC{$x \in A \times A$}
\RightLabel{[cart-mem]}
\UnaryInfC{$x.1 \in A \wedge x.2 \in A$}
\RightLabel{[$\exists-$elim$^{[2]}$]}
\BinaryInfC{$x.1 \in A \wedge x.2 \in A$}
\RightLabel{[$\wedge-$elim]}
\UnaryInfC{$x.1 \in A \qquad x.2 \in A$}
\RightLabel{[cart-mem]}
\UnaryInfC{$(x.1, x.2) \in A \times A$}
\RightLabel{[dot-elim]}
\UnaryInfC{$x \in A \times A$}
\RightLabel{[$\forall-$intro$^{[1]}$]}
\UnaryInfC{$\forall x : \operatorname{id} A \bullet x \in A \times A$}
\RightLabel{[subset]}
\UnaryInfC{$\operatorname{id} A \subseteq A \times A$}
\end{prooftree}
$$

Example 31 It is possible to prove:

$$
\frac{}{(\operatorname{id} A) \operatorname{proj} 1 \ = A}\ \text{[id-proj1]}
$$

as follows:

$$
\dfrac{
  \dfrac{\lceil y \in A \rceil^{[3]} \quad \lceil y \in A \rceil^{[3]}}{(y, y) \in \mathrm{id}\, A}\,[\text{cart-mem}]
  \qquad
  \dfrac{\dfrac{}{y = y}\,[\text{eq-ref}]}{y = (y, y).1}
}{
  \dfrac{\exists x : \mathrm{id}\, A \bullet y = x.1}{\dfrac{y \in \{x : \mathrm{id}\, A \bullet x.1\}}{\forall y : A \bullet y \in \{x : \mathrm{id}\, A \bullet x.1\}}\,[\forall\!-\!\text{intro}^{[3]}]}\,[\text{compre}]
}\,[\exists\!-\!\text{intro}]
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\lceil y \in \{x : \mathrm{id}\, A \bullet x.1\} \rceil^{[1]}}{\exists x : \mathrm{id}\, A \bullet y = x.1}\,[\text{compre}]
      \quad
      \dfrac{\lceil y = x.1 \rceil^{[2]} \quad \dfrac{\dfrac{\dfrac{\dfrac{x \in \mathrm{id}\, A}{(x.1, x.2) \in \mathrm{id}\, A}\,[\text{dot-intro}] \quad \dfrac{}{\mathrm{id}\, A \subseteq A \times A}\,[\text{id-sub-cart}]}{(x.1, x.2) \in A \times A}\,[\text{subset-mem}]}{x.1 \in A \wedge x.2 \in A}\,[\text{cart-mem}]}{x.1 \in A}\,[\wedge\!-\!\text{elim1}]}{y \in A}\,[\text{eq-sub}]
    }{y \in A}\,[\exists\!-\!\text{elim}^{[2]}]
  }{\forall y : \{x : \mathrm{id}\, A \bullet x.1\} \bullet y \in A}\,[\forall\!-\!\text{intro}^{[1]}]
  \qquad
  \forall y : A \bullet y \in \{x : \mathrm{id}\, A \bullet x.1\}
}{
  \dfrac{(\forall y : \{x : \mathrm{id}\, A \bullet x.1\} \bullet y \in A) \wedge (\forall y : A \bullet y \in \{x : \mathrm{id}\, A \bullet x.1\})}{\dfrac{\{x : \mathrm{id}\, A \bullet x.1\} = A}{(\mathrm{id}\, A)\ \mathrm{proj}\ 1\ = A}\,[\text{abbrev}]}\,[\text{ext}]
}\,[\wedge\!-\!\text{intro}]
$$

Example 32 It is possible to prove:

$$
\dfrac{}{(\mathrm{id}\, A)\ \mathrm{proj}\ 2\ = A}\,[\text{id-proj2}]
$$

analogously as we proved the previous predicate.

Example 33 It is possible to prove that

$$
\dfrac{x = y \qquad x \in A}{(x, y) \in \mathrm{id}\, A}\,[\text{id-eq1}]
$$

as follows:

$$\dfrac{\dfrac{\dfrac{x \in A \qquad \dfrac{x = y \qquad \dfrac{}{(x,y) = (x,y)}\,[\text{eq-ref}]}{(x,y) = (x,x)}\,[\text{eq-sub}]}{\exists z : A \bullet (x,y) = (z,z)}\,[\exists-\text{intro}]}{(x,y) \in \{z : A \bullet (z,z)\}}\,[\text{compre}]}{(x,y) \in \operatorname{id} A}\,[\text{abbrev}]$$

and the reverse direction:

$$\dfrac{\dfrac{\dfrac{(x,y) \in \operatorname{id} A}{(x,y) \in \{z : A \bullet (z,z)\}}\,[\text{abbrev}]}{\exists z : A \bullet (x,y) = (z,z)}\,[\text{compre}] \qquad \dfrac{\dfrac{\dfrac{\lceil (x,y) = (z,z) \rceil^{[1]}}{x = z \wedge y = z}\,[\text{cart-eq}]}{x = y}\,[\wedge\text{–elim,eq-sym,eq-trans}] \qquad \begin{array}{c} \dfrac{\dfrac{\dfrac{\lceil (x,y) = (z,z) \rceil^{[1]}}{x = z \wedge y = z}\,[\text{cart-eq}]}{x = z}\,[\wedge\text{–elim,eq-trans}] \qquad \lceil z \in A \rceil^{[1]}}{x \in A}\,[\text{eq-sub}] \\ \vdots \\ x \in A \end{array}}{x = y \wedge x \in A}\,[\wedge-\text{intro}]}{x = y \wedge x \in A}\,[\exists-\text{elim}^{[1]}]$$

Example 34 Analogously as in the previous example, we are able to prove that:

$$\begin{array}{c} x = y \qquad\qquad y \in A \\ \hline\hline (x,y) \in A \end{array}\,[\text{id-eq2}]$$

Example 35 It is possible to prove:

$$\dfrac{(x,y) \in \operatorname{id} A}{(y,x) \in \operatorname{id} A}\,[\text{id-rev}]$$

as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{(x,y) \in \mathrm{id}\, A}{x = y \land x \in A}\,[\text{id-eq1}]}{x = y}\,[\land\text{−elim1}]}{y = x}\,[\text{eq-ref}] \qquad \dfrac{\dfrac{\dfrac{(x,y) \in \mathrm{id}\, A}{x = y \land x \in A}\,[\text{id-eq1}]}{x = y}\,[\land\text{−elim1}] \quad (x,y) \in \mathrm{id}\, A}{(x,x) \in \mathrm{id}\, A}\,[\text{eq-sub}]}{(y,x) \in \mathrm{id}\, A}\,[\text{eq-sub}]
$$

3.1 Couples as bound variables

For convenience, we can define similar abbreviation for using couples instead of single identifier as we did in Section 2.4.

Suppose that p and q are predicates with free variables x and y.

$$
\begin{array}{rcl}
\{(x,y) : A \mid p\} & \equiv & \{c : A \mid p[c.1/x, c.2/y]\} \\
\{(x,y) : A \bullet q\} & \equiv & \{c : A \bullet q[c.1/x, c.2/y]\} \\
\{(x,y) : A \mid p \bullet q\} & \equiv & \{c : A \mid p[c.1/x, c.2/y] \bullet q[c.1/x, c.2/y]\}
\end{array}
$$

4 Numbers

The following rules (taken from [2, §4.1]) are related to numbers.

$$
\dfrac{x = y}{S(x) = S(y)}\,[\text{Q1}] \qquad \dfrac{}{S(x) \neq 0}\,[\text{Q2}] \qquad \dfrac{x \neq 0}{\exists x : \mathbb{N} \bullet x = S(x)}\,[\text{Q3}]
$$

$$
\dfrac{}{x + 0 = x}\,[\text{Q4}] \qquad \dfrac{}{x + S(y) = S(x + y)}\,[\text{Q5}] \qquad \dfrac{}{x.0 = 0}\,[\text{Q6}]
$$

$$
\dfrac{}{x.S(y) = x.y + x}\,[\text{Q7}] \qquad \dfrac{x \leq y}{\exists v : \mathbb{N} \bullet v + x = y}\,[\text{Q8}] \qquad \dfrac{x < y}{\exists v : \mathbb{N} \bullet S(v) + x = y}\,[\text{Q9}]
$$

The above rules (more or less) form so called Robinson's arithmetic. The Peano arithmetic contains an additional rule (actually a schema representing infinite number of rules—one for each predicate $P_{\_}$)

$$\frac{P(0) \quad P(x) \Rightarrow P(S(x))}{P(x)}\,[\mathrm{Ind}]$$

The more general form of the above inference rule schema could be (if predicate contains also some other free variables $\underline{y} = y_1, \ldots, y_n$ besides x) as follows:

$$\frac{P(0, \underline{y}) \quad P(x, \underline{y}) \Rightarrow P(S(x), \underline{y})}{P(x, \underline{y})}\,[\mathrm{Ind}]$$

It is quite easy to prove that

$$\frac{x \neq y}{S(x) \neq S(y)}\,[\neg - q5]$$

The proof of $\downarrow$ direction:

$$\cfrac{\cfrac{\cfrac{\cfrac{\lceil S(x) = S(y) \rceil^{[1]}}{x = y}\,[\mathrm{Q1}] \qquad x \neq y}{false}\,[\neg - \mathrm{elim}]}{\neg(S(x) = S(y))}\,[\neg - \mathrm{intro}^{[1]}]}{S(x) \neq S(y)}\,[\mathrm{abbrev}]$$

The proof of $\uparrow$ direction:

$$\cfrac{\cfrac{\cfrac{\lceil x = y \rceil^{[1]}}{S(x) = S(y)}\,[\mathrm{Q1}] \qquad S(x) \neq S(y)}{\neg(x = y)}\,[\neg - \mathrm{intro}^{[1]}]}{x \neq y}\,[\mathrm{abbrev}]$$

Example 36 The proof-tree of the formula $0 + y = y + 0$ looks as follows:

$$
\dfrac{\dfrac{\dfrac{}{0+S(y)=S(0+y)}\,[\text{Q5}]}{S(0+y)=0+S(y)}\,[\text{eq-sym}] \qquad \dfrac{\dfrac{}{0+S(y)=S(0+y)}\,[\text{Q5}] \quad \dfrac{\lceil 0+y=y+0 \rceil^{[1]}}{S(0+y)=S(y+0)}\,[\text{Q1}]}{0+S(y)=S(y+0)}\,[\text{eq-sub}]}{}\,[\text{eq-sub}]
$$

$$
\dfrac{\dfrac{\dfrac{}{0+0=0}\,[\text{Q4}] \quad \dfrac{}{0+0=0}\,[\text{Q4}]}{0+0=0+0}\,[\text{eq-sub}] \qquad \dfrac{\vdots}{S(0+y)=S(y+0)}\,[\text{eq-sub}]}{0+y=y+0}\,[\text{Ind}]
$$

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[1] Jim Woodcock and Jim Davies. *Using Z: specification, refinement, and proof.* Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1996.

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